MATHEMATICAL INTRODUCTION
TO
ECONOMICS

BY

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TO

ISABEL JOHN EVANS

PREFACE

In this "Mathematical Introduction to Economics" I do not attempt a voluminous or complete treatise, but give a short unified account of a sequence of economic problems by means of a few rather simple mathematical methods. I have included numerous exercises, with the purpose of making the text both available for the classroom and suitable for independent study; and the reader should find them useful not only for practice in the mathematical methods but also to complete and extend the theory. For example, in a large part of the text, the demand functions are taken as linear approximations, in which form they lend themselves to numerical calculation; but in the exercises, which may be worked by the same methods, they may be considered more generally, so that different regimes of production may be compared in theory.

I hope that I have used the minimum of mathematical technique, so that the book will be available to economists who are interested in a mathematical treatment; and also that I have used the minimum of technical economic terminology, so that persons, such as scientists and engineers, who have mathematical training and wish to know something about economics may find the text a ready introduction to the subject—easier, more penetrating, and more suggestive than the traditional exposition. The methods are the fundamental methods of the differential calculus.

For the mathematical method in general, after the valiant and successful efforts of my predecessors, I do not feel that any apology is necessary. I have accordingly deferred a general discussion of the place of mathematics in economics until Chap. X.

The text should be available in class for students of the third and fourth undergraduate years, and provide a field of application of the calculus off the beaten track, which is nevertheless of general interest. It can be covered in a half year. It does not involve any theory of statistics, but treats the kind of laws that may arise from statistical investigations and the consequences that may be deduced from them. Hence it may serve as a useful complement to a course or half course in that popular subject.

As Appendix I is printed a short bibliography of collateral reading in English or in English translation, such as in a short space will be of most service to the reader. In addition it is well to mention Pareto's now classic "Manuel d'Economie Politique," 2d ed., Paris, 1927; Amoroso's "Lezioni di Economia Matematica," Bologna, 1921; and Divisia's "Economique Rationnelle," Paris, 1928. Extensive recent bibliography is given in the concluding paragraphs of the various chapters of Pietro-Tonelli's "Traité d'Economie Rationnelle," the French translation by Gambier, Paris, 1927. Mention should also be made of H. L. Moore's "Synthetic Economics," which appeared while this book was in press.

Much of the material now given has arisen from courses of lectures which I have had the pleasure of giving at the Rice Institute in 1920 to 1921, and subsequently; at the University of California in the summer sessions of 1921 and 1928; and at the University of Chicago in the summer quarter of 1925. Some of it is based on my papers in the *American Mathematical Monthly* and in the *Proceedings* of the National Academy of Science. In particular, I am greatly obliged to the editor of the latter journal for permission to reprint in substance the material given in Appendix II. But much of the material of the book is new in the present writing. A further development of some of it, especially with reference to economic dynamics, may be found in my essay "Stabilité et Dynamique de la Production dans l'Economie Politique," now in press, as a Mémorial des Sciences Mathématiques, in the series edited by Professor H. Villat.

GRIFFITH C. EVANS.

HOUSTON, TEXAS,
March, 1930.

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MATHEMATICAL INTRODUCTION TO ECONOMICS

CHAPTER I

AN ELEMENTARY THEORY OF MONOPOLY

1. Introduction.—Although it may be our ultimate purpose to formulate general systems of theoretical economics, we must limit ourselves for the present to the consideration of special problems, and thus get an idea of the desirable elements for more extended investigations. With this point of view, we shall commence by making special hypotheses and theories for three central economic situations, namely, monopoly, cooperation or association, and competition.

2. The Cost Function.—Let us assume for the present that the cost of producing a quantity of goods in the unit of time depends solely on the amount of the commodity which is produced. Denote then the cost of manufacturing and getting to market u units of this commodity per unit time by $q(u)$. What kind of a function is $q(u)$?

One characteristic of the cost function is that the cost for excessive amounts of production is prohibitive, that is to say, that after a certain amount of production the cost rises more and more steeply with the amount produced. The cost of producing u units may for a while increase slowly with u, but ultimately increases more and more rapidly. The typical cost curve will then be assumed to be in the form of one of the following graphs (Figs. 1 and 2).

At an arbitrary point P of the graph, the ratio of the two lengths MP/OM (which is the slope of the line OP) is the quantity $q(u)/u$, which is the average cost per unit when u units are produced.

$$\text{Average unit cost} = \frac{q(u)}{u} = \text{slope of } OP. \qquad (1)$$

In both of the given figures this average unit cost, as u is increased, for a while decreases and ultimately increases again, as P pro-

ceeds far enough out on the curve so that OP begins to swing upward.

There is a second kind of unit cost which economists consider which is again interesting mathematically. It is called the marginal unit cost. Suppose that u units of a commodity are

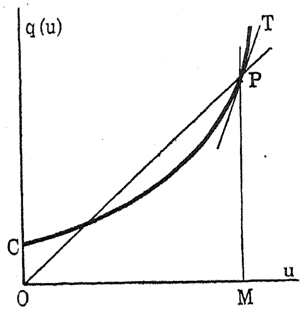

FIG. 1.

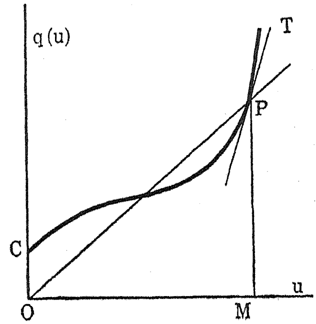

FIG. 2.

being produced, and that then the production is increased by an arbitrary small amount Δu. Then the cost will be increased by a small amount Δq illustrated in Fig. 3, and the average unit cost of producing this additional amount will be $\Delta q/\Delta u$, which is the slope of the chord PP' in the figure. This is the quantity which has the derivative dq/du as its limiting value, as the additional amount produced is taken indefinitely small.

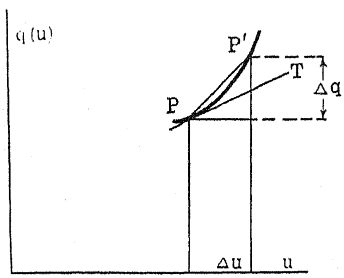

FIG. 3.

The marginal unit cost will be defined as the value of this derivative. Roughly speaking, it is the cost of increasing the production by one additional unit, when u units are already being produced per unit time. The first step in the calculus tells us that this quantity is represented by the slope of the tangent PT to the cost curve.

$$\text{Marginal unit cost} = \frac{dq}{du} = \text{slope of } PT. \tag{2}$$

In Fig. 1, the marginal unit cost continually increases as u increases, since the tangent to the curve at P gets continually steeper as P moves along the curve to the right. In Fig. 2 on the other hand the marginal unit cost at first decreases, but ultimately increases again.

Finally, we notice that the distance OC has a significant interpretation. In fact it is the cost of producing zero units of the commodity per unit time, a quantity which will in general not be zero since in it are included rents, interest on capital, etc. It is commonly called overhead expense.

These quantities may now be given analytically, if we first assume a function for the cost function. The essential character of Fig. 1 can be represented by a quadratic function.

$$Au^2 + Bu + C = q(u), \tag{3}$$

while the extra quirk in Fig. 2 would require a cubic function

$$Ru^3 + Au^2 + Bu + C = q(u) \tag{4}$$

However, for most practical problems we should be interested in a relatively small portion of the curve, which might for those values of u be well represented by a quadratic function with properly chosen values of the coefficients. Indeed as a first approximation we may well limit ourselves to a quadratic function for the whole curve, and determine the coefficients A, B, C statistically for our particular problem as well as we can.

In terms of (3) then, the overhead cost per unit time is obtained by putting $u = 0$ in (3). The overhead cost has therefore the value C. We shall want to assume C to be some positive quantity. The average unit cost is given by

$$\frac{q(u)}{u} = Au + B + \frac{C}{u}$$

which becomes large when u is small. The marginal unit cost is

$$\frac{dq}{du} = 2Au + B$$

In order for this to be large when u is large we must assume A to be a positive quantity. In fact A must be positive in order to have the curve concave upward, as in Fig. 1. We may also assume that B is positive since it is the marginal unit cost when $u = 0$ that is, roughly the cost (beyond the overhead) of producing the first unit.

EXERCISE.—Show that if the cost curve is given by (4), in order to represent a curve like Fig. 2, the coefficients must satisfy the conditions

$$R > 0,\ A < 0,\ B > 0,\ C > 0.$$

What must be said about the coefficients in order that the marginal unit cost should always be positive?

Suggestion.—State the condition that the roots of $dq/du = 0$ shall not be real, that is, that there shall be no horizontal tangents on the curve.

3. The Demand Function.—As an assumption corresponding to the one which we made about the cost function, let us suppose that the amount of goods which may be purchased in the market in unit time depends merely on the price of that commodity. We may denote by y the amount of the commodity which would be purchased under given conditions. Our assumption then is that y is a function of p alone, so that our problems may be

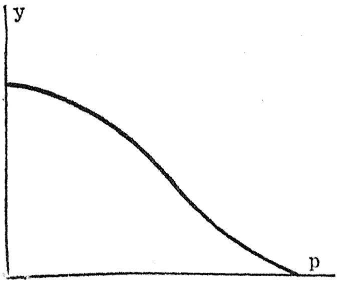

FIG. 4.

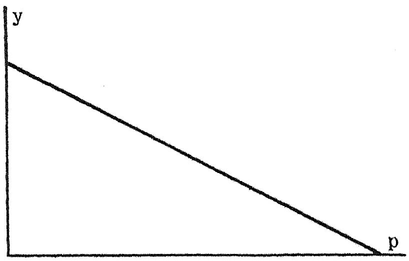

FIG. 5.

stated in terms of functions of a single variable. The typical function would then be one like that illustrated in Fig. 4, of which the characteristic property is that y decreases as p increases. In other words dy/dp is negative at all points of the curve.

The simplest form that we can give to the graph of y, which is called the demand at price p, is that of a straight line. We shall then as a first approximation assume the law

$$y = ap + b \tag{5}$$

with $a < 0$, $b > 0$ as illustrated in Fig. 5, the quantity a being the slope of the line, and b being the intercept on the y-axis. Of course any function like that in Fig. 4 can be represented for a short distance by a straight line with negative slope and positive intercept, so that (5) gives us a good approximation, for a narrow range of prices, to the more general function.

Just as in the case of the cost function, the assumption that the demand is a function of the price alone, and in particular a linear function, is merely descriptive of a class of economic situations, and does not involve any statement that this is the only class worth considering. We shall in fact consider other possi-

bilities later in more detail. But it is opportune to notice at this point that the demand may in such other cases involve the prices of other commodities, which may be possible substitutes, as well as that of the price in question; it may even involve the time; it may involve the price last week and the probable price next week or next year as well as the actual price, and it may depend on the whole history of prices. The demand for stocks on the exchange has this last character quite obviously. In fact in this case the demand not only involves this history, but depends also on the profits of the concern in question.

4. Equilibrium under Monopoly of Production.—Consider now a commodity which is manufactured by a single producer, who can set whatever price he pleases on his product, and will make only what he can sell. That is, he makes $u = y$. In order to determine the price, let us assume that the monopolist fixes it, or, what is the same thing in virtue of (5), fixes the amount which he produces, so as to make the profit a maximum.

Now the selling value of an amount y at price p is yp and therefore the profit is given by the formula

$$\pi = yp - q(u)$$

or, since $y = u$, by

$$\pi = up - q(u)$$

In virtue of (3) and (5) we have

$$\pi = u\left(\frac{u - b}{a}\right) - Au^2 - Bu - C$$

and in order for π to be a maximum we must have $d\pi/du = 0$. This yields the equation

$$\frac{2u}{a} - \frac{b}{a} - 2Au - B = 0,$$

a linear equation in u which has the solution

$$u = \frac{b + Ba}{2 - 2Aa}. \tag{6}$$

From (5) we have the relation

$$\frac{b + Ba}{2 - 2Aa} = ap + b,$$

which we can solve for the unknown p. We find the result

$$p = \frac{Ba + 2Aab - b}{2a(1 - Aa)}$$

or

$$p = \frac{b - 2Aab - Ba}{-2a(1 - Aa)}, \tag{6.1}$$

the denominator and numerator of the fraction in this second form being positive.

Let us return for a minute to the formula (6). There will be determined a positive value of u, that is, an answer to our problem, provided we assume that $b + Ba > 0$; the quantity B therefore must not be too large. Since B is the marginal unit cost when $u = 0$, this condition amounts merely to saying that the demand $ap + b$ must be positive if the price is made equal to the marginal unit cost for the first unit of production.

A further point should be noticed. Since in our formula for π in terms of u alone the coefficient of u^2 is $1/a - A$, which is negative, the profit curve is concave downward and the optimum value that we have obtained corresponds to a maximum and not to a minimum of the profit. The profit however is not inherently positive, and so the value which is obtained may happen to determine the algebraic maximum of a negative quantity. In such a case we should have the value of u which would give the minimum loss—a situation, of course, which is still a practical one. If now we evaluate π by substituting the value of u given by (6), we obtain the result

$$\pi = \frac{(b + Ba)^2}{-4a(1 - Aa)} - C. \tag{7}$$

The condition that there should be a profit rather than a loss can therefore be written as a condition on the "overhead" expense, namely

$$C < \frac{(b + Ba)^2}{-4a(1 - Aa)}$$

Although for our typical cost function we have $A > 0$, the results just given still apply if $A = 0$ and yield the results

$$u = \frac{1}{2}(b + Ba),\ p = -\frac{1}{2a}(b - Ba) \tag{6.2}$$

EXERCISE 1.—The reader may work out corresponding results when the cost function is of the form (4).

EXERCISE 2.—From the relation between u and p given by the demand law, the selling value up can be expressed in terms of u alone, and therefore can be plotted as a graph. By comparing

the graphs of selling value and cost, the position of maximum profit can be determined geometrically. Does the cost curve have to be convex downward in order to yield a position of maximum profit?

5. A Second Form of Monopoly.—We need not necessarily assume that the monopolist is a "profiteer." This being a theoretical science, we may make other convenient assumptions, if the consequences of the assumptions seem interesting. For instance, let us suppose again that the monopolist regulates his price so that he can sell all he produces (*i.e.*, $u = y = ap + b$) but that he produces so as to make not the profit a maximum but the function F

$$F = \pi + k^2u \tag{8}$$

a maximum, where k^2 is some positive constant fixed at pleasure.

In this case we have

$$F = pu + k^2u - Au^2 - Bu - C$$

or

$$F = pu - Au^2 - (B - k^2)u - C$$

The function which we want to make a maximum this time differs from π only in having B replaced by $B - k^2$. Hence we can obtain the values of u and p by replacing B in the old values (6), (6.1) with $B - k^2$. Thus we obtain

$$u = \frac{b + (B - k^2)a}{2 - 2Aa}, \qquad p = \frac{b - 2Aab - (B - k^2)a}{-2a(1 - Aa)} \tag{9}$$

In this way production is increased over the original value, given in (6), by the positive quantity $-k^2a/(2 - 2Aa)$, the price is diminished by the positive quantity $k^2/(2 - 2Aa)$. If, further, we calculate the profit $\pi = pu - Au^2 - Bu - C$ with the new values of u and p, we find by calculation that it is less than the profit under the "profiteering" conditions by

$$\frac{|k^4a|}{4(1 - Aa)}$$

whereas the function F, which now has its maximum value, exceeds the value which it would have had with the original p, u by the same quantity. In particular, the profit will not disappear unless k^2 is given a value sufficiently large so that the diminution of π is equal to the original value of π, given by (7), *i.e.*, unless

$$k^4 \geq \frac{(b + Ba)^2}{a^2} - \frac{4C(1 - Aa)}{-a}$$

6. Price Fixing.—In order to get a theory radically different from those so far developed, let us now make the assumption that the price is to be held constant, not depending on the amount of production u. The situation is illustrated by the graph of Fig. 6. Besides the cost curve, which is the graph of the function $q(u)$, let us draw what we may call a price line, namely a line through O whose slope is taken as the given constant price p:

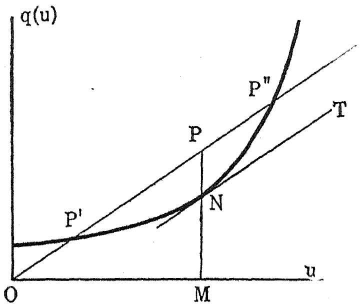

FIG. 6.

$$\tan \angle MOP = p$$

For this line the ordinate, which is the slope multiplied into the abscissa, or pu, represents the selling value of the u units of commodity. Since the cost is represented by the ordinate up to the cost curve, the profit or loss will be directly given on the graph. Thus in the special position $u = \overline{OM}$, the profit is NP, and in the cases where u is the abscissa of P' or P'' there will be neither profit nor loss; there is profit for points between P' and P'', but loss corresponding to points to the left of P' or to the right of P''.

The situation which gives a maximum profit at the price p will be that where the tangent NT is parallel to the price line, since if NT were not parallel to the price line the length NP could be increased by moving M to the right or left. But the slope of NT is dq/du, which is the marginal unit cost. Hence the optimum production is determined by the equation

$$\text{marginal unit cost} = p. \tag{10}$$

If $q(u)$ is the quadratic function $Au^2 + Bu + C$, the marginal unit cost is $2Au + B$, and in order that u shall create the maximum profit at the given price p the quantity $\pi = pu - Au^2 - Bu - C$ must be a maximum, p being held constant. Setting $d\pi/du = 0$, we obtain the relation

$$p - 2Au - B = 0$$

which is, of course, the equation (10). This determines u in terms of p:

$$u = \frac{p - B}{2A} \tag{11}$$

a quantity which we may call the offer by the producer for the price p. The offer is thus a function of the given price. In the case considered, its graph is a straight line with positive slope $1/(2A)$ cutting the p-axis B units to the right of the origin.

An interesting situation is obtained by choosing such a price that the offer at that price will be equal to the demand at that price, given by (5). We have

$$\frac{p - B}{2A} = ap + b,$$

whence

$$p = \frac{B + 2Ab}{1 - 2Aa}, \qquad \text{and} \qquad u = \frac{b + Ba}{1 - 2Aa} \tag{12}$$

by (5). This equilibrium price is illustrated by p_0 in Fig. 7. We notice that the price in (9) is less than that given by (6) and the amount produced greater than that given by (6.1). In fact, fixing the price at less than the profiteer's natural price need not drive the goods from the market; it may on the contrary, as here, actually increase the amount which the producer will desire to furnish to the market.

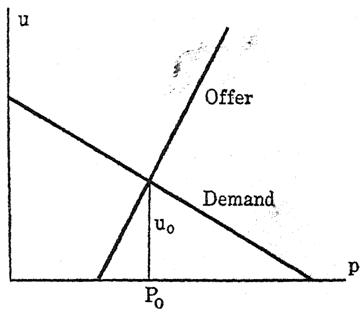

FIG. 7.

EXERCISE.—Show that the situation given by (12) is equivalent to that described in Sec. **5** by putting $k^2 = \dfrac{b + Ba}{-a(1 - 2Aa)} = \dfrac{u_0}{-a}$ (see Fig. 7). Compare this u with the value of u for which $\pi^2 - \dfrac{u^2}{a}$ is a maximum.

7. Offer and Demand.—The result of the previous section suggests the question as to whether situations can be described in which equality of offer and demand, as exemplified in Fig. 7, will determine the natural equilibrium price of a market. Consider the case where there are many producers, and each one regards his production as too small to influence the price of the commodity in the market. We may assume then that if his cost function is $A_i u^2 + B_i u + C_i$, his offer will be

$$u_i = \frac{p - B_i}{2A_i}$$

and there will be a total offer in the market of amount

$$u = \sum u_i = p\sum \frac{1}{2A_i} - \sum \frac{B_i}{2A_i},$$

which is again incidentally a linear function of p, as in Fig. 7, and can be written as $u = \alpha p + \beta$. But in order for the market to be in equilibrium the amount brought to the market must equal the amount which is bought in the market in the same time. In other words, the total offer is equal to the demand, and the price and the total amount produced will be determined as in Fig. 7, where α takes the place of $1/2A$ and β of $-B/2A$.

If the total offer is less than the demand, the situation is one in which $p < p_0$; we may assume that the price will rise. On the other hand if the offer is greater than the demand and therefore $p > p_0$, we may assume that the price will fall. In other words, the equilibrium price determined by the offer-demand diagram is one towards which the price tends, or across which it will oscillate, when the market is not in equilibrium.

It must be emphasized, in resumé, that it is only in such special cases that this conventional offer-demand diagram is applicable, a result which will become still more evident in Chapter III. We have already seen that it is not applicable in the strict monopoly problem.

EXERCISE.—In the plane of Fig. 6, assumed to represent an individual cost curve, find the coördinates of P' and P'', that is, the intersections of the line $\alpha = pu$ and the curve $\alpha = Au^2 + Bu + C$. In order that P' and P'' coincide, and thus the optimum position be one neither of profit nor loss, show then that $(B - p)^2 = 4AC$ and that $p = B + 2\sqrt{AC}$.

From the fact that for this position marginal unit cost equals average unit cost show that $u = \sqrt{C/A}$.

8. General Exercises.

1. Show that at a value of u for which the average unit cost is a minimum, it must be equal to the marginal unit cost, no matter what the cost function is.

Consider as a special case that where there is no overhead cost.

2. Show that at a positive value of u for which the marginal unit cost is a minimum the cost curve in general has a point of inflexion.

3. Obtain the price and production which make profit a maximum under the monopoly hypothesis, if the demand function is $y = f(p)$ and the cost function is $q(u)$, production being equal to demand, and these functions being arbitrary.

4. Obtain a formula for the offer as a function of p when the cost curve is given by the cubic function (4). Can you eliminate all but one possible solution?

5. By means of the demand law in the case of monopoly, which we have assumed, namely $y = u = ap + b$, we can express p and therefore the selling value up as a function of u. Show that the graph of the latter function is a parabola with its axis along the downward vertical, one end of the graph being at the origin; and the other at the distance b from the origin. By drawing this graph on the same figure with the cost curve, determine the monopoly production which gives the maximum profit. Explain Fig. 8. What length shows the profit? the quantity produced? the price?

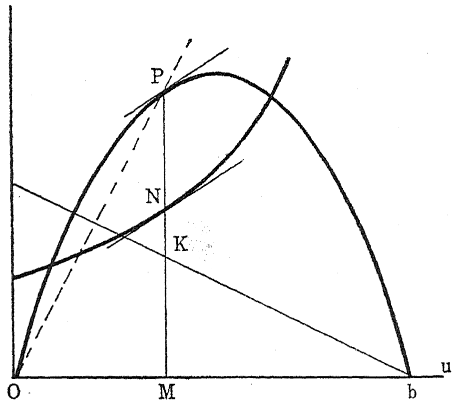

FIG. 8.

6. On Fig. 8 the dotted line OP is the price line for the equilibrium price. Why? What would be the offer at that price? Is it greater or less than OM?

7. The elasticity of demand is defined by the quantity $\eta = \dfrac{-dy/y}{dp/p} = -\dfrac{p}{y}\dfrac{dy}{dp}$ and is thus a measure of the proportional change of y due to a proportional change of p. For what demand law is η constant?

We have

$$\frac{dy}{y} = -\eta\frac{dp}{p} \text{ or } \log y = -\eta \log p + \text{constant}$$

or finally

$$yp^{\eta} = k$$

where k is an arbitrary constant.

8. Draw a graph of η as a function of p when $y = ap + b$. Show that in this case $d\eta/dp$ is > 0. In the figure representing the demand, $y = f(p)$, show that η is given by the ratio $\eta = MT/OM$, η being the elasticity of demand at P. (See Fig. 9.)

9. Consider the quantity

$$\xi = -\frac{1}{y}\frac{dy}{dp}$$

Since $\xi = -\frac{d}{dp}\log y$, the demand curves for which ξ is constant are

$$y = ke^{-\xi p}$$

Show that if this is the demand law and $q(u) = Au^2 + Bu + C$ there is only one solution of the monopoly problem.

10. Discuss the monopoly problem in which $\pi = py - Au^2 - Bu - C$, $y = ap + b + mu$, and $u = y$. Treat separately the cases $m < 1$ and $m > 1$. Compare graphically the cost and selling value curves.

11. Consider the monopoly problem in which $q = Au^2 + Bu + C + z$, z being a given cost of advertising. Suppose that the effect of advertising is to change the demand from the form $y = ap + b$ to the form $y = ap + b + kz$, $k > 0$, and take $u = y$. Show that if an advantage of profit is possible with a given expense of advertising z it will be increased by increasing that expense.

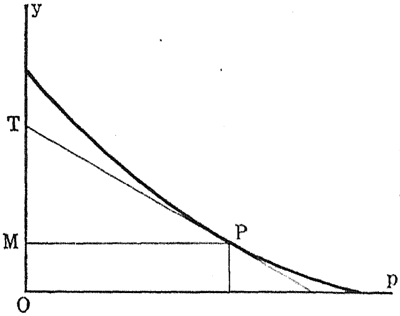

FIG. 9.

Can the equilibrium values u and p both increase if z is increased? Can u increase and p decrease, if z is increased? Consider the effect of the algebraic sign of A.

12. Suppose that a commodity is sold by means of agents or drummers who add to the cost function $q = Au^2 + Bu + C$ an additional term ξ, and to the demand $y = ap + b$ an additional term η, proportional to the number of such agents, that is, $\eta = k\nu$.

We may make various hypotheses about ν and ξ. For example, we may assume that ν = constant, or that ν is proportional to the amount which is sold ($k\nu = my$) or is a linear function of that amount ($k\nu = my + g$). We may assume that the cost of the drummers is merely a wage ($\xi = \alpha\nu$), or is a commission on the value which is sold, all of the production or merely a part of it being sold in this way ($\xi = hup$) or is a combination of the two methods.

Find the price p and production u for the monopoly problem in which $y = u$, and $\pi = py - q$ is to be made a maximum as a function of p or u.

CHAPTER II

ON CHANGE OF UNITS

9. Dimension.—The cost function is merely an end result of the process of cost accounting. But the various items that appear are usually given in terms of different kinds of units of money, quantity and time—cents and dollars, pounds of material, and tons, wages per week, rent per year, cost per unit of material, etc. Suppose then, choosing definite units, that it has been possible to build up a cost function finally out of all these items—that being, as we have defined it, the cost of so-much commodity per unit of time, and perhaps a complicated function—and suppose that it is desirable in the completed cost function to change the units; of time, from the week to the year, say; of quantity, from the pound to the ton; and of money from the dollar to the thousand dollars, or to some unit of foreign currency such as the franc. Will it be necessary to re-examine all the items out of which the cost function has been built? The answer to this question is furnished in terms of the dimensions of the various units involved.

We consider, as an example, an illustration from another science; and take the simple case of uniform speed, which is defined as the number of units of distance traversed in a time t divided by the number of units of the time t. The unit of speed is the speed for which the distance and the time are both measured by the same number of units, or, in particular, that for which both distance and time are one unit:

$$v = \frac{s}{t} \tag{1}$$

$$v = 1$$

when

$$s = t,$$

in particular, when

$$s = t = 1.$$

If we change to units of different amounts for s and t, the unit for v will also change, and in a precise manner. Suppose that

v is measured in new units, that for s being k times and that for t being n times the old; let the measures in the new units be indicated by primed letters. Then, by (1), $v' = 1$ when $s' = t' = 1$: that is $v' = 1$ when $s = k$, $t = n$ or when $v = k/n$. Hence

$$\text{unit of } v' = \frac{k}{n}(\text{unit of } v) \tag{2}$$

and the measure v' of the speed in the new units in n/k times the measure v in the old units.

$$v' = \frac{n}{k}v. \tag{3}$$

In theoretical mechanics, it is assumed that all the units may be treated in this way in terms of three fundamental units, which are usually chosen as those for mass, distance, and time.

In the subject to which these lectures are devoted we need a unit M of value in money, units $J_1, J_2, \ldots J_k$, of quantities of the various commodities involved, and a unit T of time. For the present we shall use merely M, J, T, restricting ourselves as in Chapter I to a single commodity.

Two quantities (or their units) are said to be of the same dimension if their ratio remains unchanged no matter how the fundamental units are changed. More exactly, if the fundamental units are multiplied by numbers l, k, n the units of the two quantities under consideration will be multiplied by certain two numbers α, β respectively; if $\alpha = \beta$ whatever the l, k, or n, the two quantities (or units) are of the same dimension with respect to the fundamental quantities (or units). In particular, if

$$\alpha = l^r k^s n^v$$

we say that the first quantity is of dimension r with respect to the first fundamental quantity, of dimension s with respect to the second, and of dimension v with respect to the third. If the quantity is x, we represent the dimension in terms of the symbolic statement,

$$[x] = M^r J^s T^v. \tag{4}$$

The dimensions r, s, v may happen to be negative as well as positive.

From the analysis of the example of uniform speed, given above, it is evident that uniform speed is of dimension $+1$ in length and -1 in time; since, when the unit of length is multiplied

by k and the unit of time by n, the unit of speed is multiplied by $k^{+1}n^{-1}$.

What we call a measurable quantity, or simply a quantity, is one which can be measured in terms of a unit, and its measure is its ratio to the particular unit; this ratio is assumed to be determinate and is the same as the ratio of the measures of the quantity and the unit respectively, in terms of any new unit. Hence, in general, if the unit is multiplied by a factor, the measure in terms of the new unit will be the original measure divided by the factor. What we have said so far can be reassumed in terms of the following statement.

If the unit M is multiplied by l, the unit J by k, and the unit T by n, then if $[x] = M^rJ^sT^v$, the unit of x will be multiplied by $l^rk^sn^v$,

$$X' = l^rk^sn^vX \tag{5}$$

and the measure x' of x in the new units will be its measure x in the old units, divided by the same factor:

$$x' = l^{-r}k^{-s}n^{-v}x \tag{5.1}$$

10. Theorems on Dimensions.—We notice that if equation (5.1) holds, when the fundamental units are multiplied by l, k, and n respectively, it follows that $x' = 1$ when $x = l^rk^sn^v$. Hence

$$[x] = M^rJ^sT^v.$$

We have then, recalling also equation (5.1), a first theorem and corollary.

THEOREM 1.—*A necessary and sufficient condition that*

$$[x] = M^rJ^sT^v$$

is that

$$x' = l^{-r}k^{-s}n^{-v}x.$$

COROLLARY.—*A necessary and sufficient condition that x shall be of dimension r in the fundamental unit M, even if x depends on the other units, is that $x' = l^{-r}x$, when the unit M is multiplied by l and the other units are unchanged.*

We may also establish directly the following statements:

THEOREM 2.—*If* $[x] = M^{r_1}J^{s_1}T^{v_1}$

and

$$[y] = M^{r_2}J^{s_2}T^{v_2}$$

then

$$[xy] = M^{r_1+r_2}J^{s_1+s_2}T^{v_1+v_2}$$

$$\left[\frac{x}{y}\right] = M^{r_1-r_2}J^{s_1-s_2}T^{v_1-v_2}.$$

If by a change of unit in one fundamental quantity, say time, we have

$$x' = n^{-v_1}x$$
$$y' = n^{-v_2}y,$$

we shall have

$$x'y' = n^{-(v_1+v_2)}xy.$$

Hence by the Corollary to Theorem 1, the dimension of $x'y'$ in T is $v_1 + v_2$. A similar proof applies to the other units and to division.

Exercise.—Show that if x and y are of the same dimensions, the quantity $x + y$ is of the same dimensions. If x and y are not of the same dimensions, $x + y$ will not be a quantity in the sense of Sec. 9.

Relations with which we deal will generally involve variable quantities. According to the idea of quantity expressed in Sec. 9, the dimension of the quantity in any specified fundamental unit remains invariant; it is the measure of the quantity which changes. Consider then relations of the form:

$$y = f(x, C_1, C_2, \cdots C_p), \tag{6}$$

over a range of variables in x, where x, y and the C's have given dimensions:

$$[x] = M^rJ^sT^v,\ [C_i] = M^{r_i}J^{s_i}T^{v_i},$$
$$[y] = M^{r'}J^{s'}T^{v'}.$$

If we have a sequence of such functions

$$y_q = f_q(x, C_1, \cdots, C_p),\ q = 1, 2, \cdots,$$

all of the same dimension in each fundamental unit, which have a limit function,

$$y = f(x, C_1, \cdots, C_p),\ \lim y_q = y,$$

this limit function has the given dimension in each fundamental unit.

In fact, if we change a given unit J to $J' = k^sJ$, in the measure of x, we have

$$y_q' = f_q(x', C_1', \cdots, C_p'),\quad x' = k^{-s}x,$$

which, by hypothesis, takes the form

$$y_q' = k^{-s'}y_q$$

But limit $y_q' = y'$, and limit $y_q = y$, by hypothesis. Hence

$$y' = k^{-s'}y$$

and y is of dimension s' in J. Similarly for the other units.

From this, we have immediately an important theorem:

Theorem 3.—*If y is of the form* (6), *and dy/dx exists, then*

$$\left[\frac{dy}{dx}\right] = M^{r'-r}J^{s'-s}T^{v'-v}.$$

In fact, by Theorem 2 and the Exercise,

$$\left[\frac{\Delta y}{\Delta x}\right] = \left[\frac{y_2 - y_1}{x_2 - x_1}\right] = M^{r'-r}J^{s'-s}T^{v'-v}.$$

But the limit of this function as Δx approaches 0 arbitrarily, or over a sequence of values, is precisely dy/dx. Hence the theorem is proved.

Incidentally, we notice that this theorem covers the case of partial differentiation, since that is merely differentiation where all the variables but one are held constant.

This theorem enables us to calculate the dimensions of such things as instantaneous speed, acceleration, marginal unit cost, etc. In fact, since

$$\text{speed} = \frac{dx}{dt} \text{ where } x = \text{distance},$$

we have for the dimensions of speed

$$[\text{speed}] = XT^{-1},$$

which are the same as those of uniform speed.

Exercise 1.—Show that acceleration has dimension 1 in distance and -2 in time. By what factor is the measure of acceleration multiplied if the unit of distance is changed from the foot to the inch, and the unit of time from the second to the minute?

Exercise 2.—If y is of the form (6), show that $[\int y dx] = M^{r'+r}Q^{s'+s}T^{v'+v}$.

11. Dimensions of Economic Quantities.—The symbol u has been used to denote quantity produced in unit time, and we have so far been dealing with a uniform rate of production. Hence by Theorem 2

$$[u] = JT^{-1}, \tag{7}$$

so that u is of dimension 1 in quantity and -1 in time. We have still the same result if the production is not uniform, if we let u be the rate of production, *i.e.*—the derivative of the quantity produced with respect to the time. The result follows in this case from Theorem 3.

Similarly, the symbol $q(u)$ denotes cost per unit time, if this rate is uniform with respect to time; more generally we may

regard $q(u)$ as the instantaneous rate, if it does not happen to be uniform. In any case, by Theorems 2 and 3

$$[q(u)] = \frac{[\text{money value}]}{[\text{time}]} = MT^{-1}$$

In the more general case we might of course regard q as a function of t as well as u.

Since in the cost function $q(u) = Au^2 + Bu + C$ the various quantities Au^2, Bu and C are added they must all be of the same dimension, and of the dimension of $q(u)$. Hence

$$[A][u^2] = MT^{-1}$$
$$[A]J^2T^{-2} = MT^{-1}$$
$$[A] = MJ^{-2}T = \frac{MT}{J^2}. \tag{8}$$

Similarly

$$[B] = MJ^{-1}, [C] = MT^{-1}. \tag{8.1}$$

Again, the selling value, per unit time, of goods produced at the rate u is pu, and like the cost function, is of dimensions MT^{-1}. Hence

$$[p]\,[u] = MT^{-1}$$

and, since $[u] = JT^{-1}$,

$$[p] = MJ^{-1}. \tag{9}$$

Also ap and b are of the same dimensions as u; whence

$$\begin{aligned} [a]\,[p] &= JT^{-1} = [b]. \\ [a] &= J^2M^{-1}T^{-1},\ [b] = JT^{-1}. \end{aligned} \tag{10}$$

The marginal unit cost is dq/du and the average unit cost is q/u; therefore both are of dimensions $MT^{-1}/JT^{-1} = MJ^{-1}$,

$$\left[\frac{dq}{du}\right] = \left[\frac{q}{u}\right] = MJ^{-1}. \tag{11}$$

Exercise.—Calculate $[R]$ if $q(u)$ is given by the cubic function

$$Ru^3 + Au^2 + Bu + C.$$

12. Statement of Relations in Economics.—Theoretical relations in economics are susceptible of being expressed by equations in terms of quantities of the sort described in Sec. **9**; that is quantities of definite dimensions. Moreover the relations must not depend essentially on the way the quantities are measured, in the sense that it must be possible to write them in such form that they are valid whatever be the choice of magnitudes for the

fundamental units. In this form both members will have the same dimensions in each of the fundamental units, and they will remain valid, however the units are changed.

Thus the familiar equation for a falling body

$$s = (16.1)(t^2)$$

presupposes that s is measured in feet and t in seconds; but it may be written also in the form

$$s = \tfrac{1}{2}gt^2$$

where the constant g has dimensions 1 in length and -2 in time. Knowing the value of the constant in one set of units we are able to calculate it immediately in any other by means of a formula similar to (5.1).

EXERCISE.—Given $g = 32.2$ when the units are feet and seconds, calculate g when the units are centimeters and seconds, 1 foot being 30.5 centimeters.

The requirement, stated above, is satisfied by the relations which we have met with in Chapter I. For instance, a condition that average unit cost be a minimum is that it be equal for that value of u to the marginal unit cost:

$$\frac{q(u)}{u} = \frac{dq(u)}{du}.$$

But by Theorems 2 and 3 both members of this equation are of the same dimensions. We are therefore allowed to state the following proposition, at least as a heuristic principle, or as a partial definition of economic theory:

HEURISTIC PRINCIPLE: *Laws may be written so that their formal expression may be made invariant of a change of units. Both members of any equalities or inequalities used in such laws will then have the same dimension in each of the fundamental units.*

The demand that any quantity must be measurable in terms of units seems at first glance to be trivial. For what can economists talk about which is not measurable? It is precisely this tendency to talk about not measurable things as if they were measurable which drives a fair amount of economics on the reefs, and we shall find the criterion useful. It is common, for example, to define "Wealth" as the total of material things owned by human beings, and to talk about making "pleasure" a maximum; but neither of these things has any significance in the sense of measure, and we cannot make either a maximum. They are

not quantities in the sense of Sec. **9**; they do not have dimensions. Hence they cannot be terms of relations in theoretical economics.[1]

Exercise.—If for the demand function y we have the formula

$$y = \alpha e^{-\beta p}$$

we must have $[\alpha] = [y] = JT^{-1}$, $[\beta] = [p]^{-1} = JM^{-1}$

Suggestions.—Take

$$[\alpha] = M^{r_1}J^{s_1}T^{v_1}, \; [\beta] = M^{r_2}J^{s_2}T^{v_2}$$

then

$$[\beta p] = M^{r_2+1}J^{s_2-1}T^{v_2}$$

Make a change in the unit M, multiplying it by l. Then

$$y' = l^{-r_1}\alpha e^{-\beta p l^{r_2+1}},$$

and since

$$y' = l^0 y = y$$

$$l^{-r_1}\alpha(e^{-\beta p})^{l^{r_2+1}} \equiv \alpha e^{-\beta p}$$

or

$$l^{-r_1}(e^{-\beta p})^{l^{r_2+1}-1} \equiv 1$$

from which $r_1 = 0$, $r_2 = -1$, in order for the identity to hold for all values of l and p. A still shorter method is to form first dy/dp.

It may happen that a statement which connects economic quantities may be a proper theorem without at first appearing to be so. In order to bring it into a form in which all the terms have dimensions may require some mathematical transformation. Thus, in the exercise just given both members of the equation

$$y = \alpha e^{-\beta p}$$

have dimensions, and satisfy the conditions of our heuristic principle, if $[\alpha] = [y]$ and if βp is of zero dimension in all units. A form which is however equivalent mathematically to the above is

$$\log y = \log \alpha - \beta p,$$

and in this form neither member has dimensions. In accordance with our definitions, $\log y$ is not an economic quantity, although $\log (y/\alpha)$ is one, and of zero dimensions in all units.

13. General Exercises.

1. If the unit of time is changed from the month to the year, the unit of money from the dollar to the thousand dollars, and the unit of quantity from

[1] "Wealth" may perhaps be regarded as a tensor or complex number, with one component for each kind of wealth; but the concept has not a great deal of significance, since it lacks application.

the pound to the ton, give the changes in the units for the economic quantities considered in Sec. **11,** and give the new values of those functions in terms of the old.

($[q] = MT^{-1}$, so the new unit of q is 1,000/12 times the old, and the new measure of q is 12/1,000 times the old, etc.)

2. Show that the terms in the numerator of the formula for monopoly price in Chapter I are all of the same dimensions; likewise the terms of the denominator.

3. Labor in manufacturing shoes in a given concern is supposed to be \$1,000 per week when 50 pairs of shoes are manufactured in a day. Assuming that the amount of labor is proportional to the output, consider it as the term Bu of the cost function and calculate B when the unit of quantity is the pair of shoes, of money value the dollar and of time the week. What value of u is given in the data? Change the unit of time to the day, and recalculate B. Check by the theory of dimensions.

CHAPTER III

COMPETITION AND COOPERATION

14. Two Producers.—Suppose that there are two producers, manufacturing amounts u_1 and u_2 of a given commodity in unit time. In order to simplify the situation as much as possible let us suppose that both are subject to the same cost function

$$q(u_i) = Au_i^2 + Bu_i + C, \quad i = 1, 2; \tag{1}$$

and there is produced only what is sold, so that the market remains in a steady state, or state of equilibrium. In this way our analysis will show the effect of the various combinations of production, unmasked by extraneous influences. If we assume the same demand function as in Chapter I, linear in terms of the price, we have

$$u_1 + u_2 = y = ap + b \tag{2}$$

Then the respective selling values of the amounts sold in unit time at price p are respectively pu_1 and pu_2 and the profits

$$\pi_i = pu_i - (Au_i^2 + Bu_i + C), \quad i = 1, 2. \tag{3}$$

The variables p, u_i are however not yet determined. It remains to add some arbitrary but convenient hypothesis which will completely describe the situation, and in order to do this we must consider how a function of several variables attains its maximum or minimum.

15. Maxima of Functions of Several Variables.—Suppose that $z = f(x, y)$ is a given function of x and y, and that for the particular pair of values x_0, y_0, inside the region under consideration for x and y, the function z attains its maximum value. That means, among other things, that if we hold y constant and equal to y_0, then as x varies the function $z = f(x, y_0)$, which is now a function of a single variable, will attain its maximum value when $x = x_0$. In other words, when $y = \text{const.} = y_0$, the derivative of z with respect to x must vanish at the particular value $x = x_0$. But taking things the other way round, holding x constant and equal to x_0, and letting y alone vary, the derivative of z with respect to y must vanish when $y = y_0$. What

we have said then, is that if x_0, y_0 is a pair of values for which z is a maximum, the derivative of z considered as a function of x alone and the derivative of z considered as a function of y alone must both vanish if at the same time x is put equal to x_0 and y is put equal to y_0. We have thus two equations to determine x_0 and y_0.

We denote these derivatives by the symbols $\partial z/\partial x$ and $\partial z/\partial y$, in order to emphasize the fact that in the calculation of the derivative of z with respect to x, y is considered as if it were a constant and in the calculation of the derivative of z with respect to y, x is considered as if it were constant. But these derivatives are calculated in the usual way. Thus in the function

$$z = 10 - x^2 - y^2 - xy$$

we have

$$\frac{\partial z}{\partial x} = -2x - y, \quad \frac{\partial z}{\partial y} = -2y - x$$

and the pair of values x_0, y_0 is determined by the equations

$$0 = -2x_0 - y_0$$
$$0 = -2y_0 - x_0$$

whence $x_0 = 0$, $y_0 = 0$. The corresponding value of z is $z = 10$.

In order to know if we have really a maximum, we need not only to be able to carry this process through, and find when $\partial z/\partial x$ and $\partial z/\partial y$ vanish, but also we have to decide whether we have a maximum or a minimum, or a maximum while one variable is changing and a minimum while the other is changing (a saddle point), for these situations and even others which are possible all satisfy the conditions $\partial z/\partial x = 0$, $\partial z/\partial y = 0$ when $x = x_0$ and $y = y_0$. However if we find that there is only one pair of values x_0, y_0 which satisfy these equations, and if we know from the problem that z must have a maximum, then we know the values found must be the ones we are after. This is the situation that we usually meet in our economic problems.

EXERCISE.—Show that $x_0 = 0$, $y_0 = 0$ gives a maximum value when $z = 10 - x^2 - y^2$, a minimum value when $z = 10 + x^2 + y^2$, a "saddle point" when $z = 10 + x^2 - y^2$.

16. Cooperation.—We may describe by the word cooperation or association the situation where all the producers unite in making some predetermined quantity a maximum. The simplest such quantity is the profit. We may then take as a determining postulate the following:

Each producer tries to determine the amount u_i of his production per unit time so as to make the total profit per unit time a maximum.

The total profit is given by

$$\pi = \pi_1 + \pi_2 = (u_1 + u_2)p - q(u_1) - q(u_2).$$

This is a function which varies continuously with u_1 and u_2 and never gets positively infinite (although it does get negatively infinite), hence it is bounded above by some value, and has a maximum. The situation becomes clearer perhaps if we make use of (2) and express π directly in terms of u_1 and u_2 as follows

$$\pi = (u_1 + u_2)\left(\frac{u_1 + u_2 - b}{a}\right) - A(u_1^2 + u_2^2) - B(u_1 + u_2) - 2C.$$

To find the maximum, then, we find the values of u_1 and u_2 which satisfy the equations

$$\frac{\partial \pi}{\partial u_1} = 0 = \frac{\partial \pi}{\partial u_2} \tag{4}$$

These amount to the following:

$$0 = \frac{2(u_1 + u_2)}{a} - \frac{b}{a} - 2Au_1 - B$$
$$0 = \frac{2(u_1 + u_2)}{a} - \frac{b}{a} - 2Au_2 - B.$$

Hence, by subtraction, $u_1 = u_2$, and

$$0 = \frac{(4u_1 - b)}{a} - 2Au_1 - B$$

so that

$$u_1 = u_2 = \frac{b + Ba}{4 - 2Aa}$$

From (2) we can calculate p, and if we write $u = u_1 + u_2$, we shall have

$$u = u_1 + u_2 = 2u_1 = 2u_2 = \frac{b + Ba}{2 - Aa}$$
$$p = \frac{b - Aab - Ba}{-a(2 - Aa)} \tag{5}$$

EXERCISE.—Show by comparing (5) with (6) of Chapter I, that the u just determined, is greater than the u for monopoly. What can be said about the values of p in the two cases?

Note.—The problem can still be solved, although the effect of the cooperation is not so strongly brought to view, if each producer is subject to a different cost function. In this case

$$\pi = (u_1 + u_2)\frac{u_1 + u_2 - b}{a} - A_1u_1^2 - B_1u_1 - C_1 - A_2u_2^2 - B_2u_2 - C_2$$

$$0 = \frac{2(u_1 + u_2)}{a} - \frac{b}{a} - 2A_1u_1 - B_1$$

$$0 = \frac{2(u_1 + u_2)}{a} - \frac{b}{a} - 2A_2u_2 - B_2$$

from which it no longer follows that $u_1 = u_2$. But we can solve these equations as simultaneous linear equations in u_1 and u_2 and obtain

$$u_1 = \frac{A_2b + A_2B_1a + B_2 - B_1}{2(A_1 + A_2 - A_1A_2a)}, \quad u_2 = \frac{A_1b + A_1B_2a + B_1 - B_2}{2(A_1 + A_2 - A_1A_2a)} \tag{6}$$

and from this again p may be calculated

$$p = \frac{b(A_1 + A_2) - (A_1B_2 + A_2B_1)a - 2A_1A_2ab}{-2a(A_1 + A_2 - A_1A_2a)}$$

It should be remarked that here as in the problem of Chapter I, the value of C has no influence on the values of u_i, p.

EXERCISE.—Show that this price is the same as if the coefficients of the cost functions had been the same for both producers, and had the values

$$A = \frac{2A_1A_2}{A_1 + A_2}, \quad B = \frac{A_1B_2 + A_2B_1}{A_1 + A_2}, \quad C \text{ arbitrary}$$

How about the value of $u_1 + u_2$?

17. Competition.—If, in contrast with the situation of the previous section, producers are interested each in achieving an individual object, that is, in making some quantity which refers to him specially a maximum, we shall say that we are dealing with phenomena of competition. As a possible determining postulate we may take that by means of which originally Cournot defined what he called competition.[1]

Each competitor assumes that the production of the other or others is independent of his, and tries to make his profit a maximum.

[1] Many economists speak of this situation as "monopoly of two producers," or in the corresponding more general case as "monopoly of n producers." Cournot's formulation of the theory of monopoly and of this kind of competition goes back to 1838.

The mathematical expression of this postulate is to regard u_1 and u_2 as independent variables, and write

$$\frac{\partial \pi_1}{\partial u_1} = 0, \qquad \frac{\partial \pi_2}{\partial u_2} = 0,$$

that is

$$0 = p + u_i \frac{\partial p}{\partial u_i} - q'(u_i),\ i = 1, 2, \tag{8}$$

and since, solving (2) for p and differentiating, we have $\partial p/\partial u_i = 1/a$,

$$0 = p + \frac{u_i}{a} - 2Au_i - B,\ i = 1, 2.$$

If from these equations we eliminate u_1 and u_2 by adding them together and then making use of (2), we shall have a single equation which we may solve immediately for p, and obtain finally

$$p = \frac{b - 2Aab - 2Ba}{-a(3 - 2Aa)},\ u_1 = u_2 = \frac{b + Ba}{3 - 2Aa}$$
$$u = u_1 + u_2 = \frac{b + Ba}{\frac{3}{2} - Aa} \tag{9}$$

Exercise.—As in the note to the previous section show that the present problem is solvable when the coefficients of the cost functions are not the same for the two producers, and that

$$u_1 = \frac{b - 2A_2ab - 2A_2B_1a^2 + (2B_1 - B_2)a}{3 - 4a(A_1 + A_2) + 4A_1A_2a^2},$$
$$u_2 = \frac{b - 2A_1ab - 2A_1B_2a^2 + (2B_2 - B_1)a}{3 - 4a(A_1 + A_2) + 4A_1A_2a^2}$$
$$u = u_1 + u_2 = \frac{2b - 2(A_1 + A_2)ab - 2(A_1B_2 + A_2B_1)a^2 + (B_1 + B_2)a}{3 - 4a(A_1 + A_2) + 4A_1A_2a}$$
$$p = \frac{-b + 2(A_1 + A_2)ab - 2(A_1B_2 + A_2B_1)a^2 - 4A_1A_2a^2b + (B_1 + B_2)a}{a[3 - 4a(A_1 + A_2) + 4A_1A_2a^2]}$$

Show that these formulae reduce to those of Sec. **17** if $A_1 = A_2$, $B_1 = B_2$.

18. A Second Kind of Competition.—It is not difficult to think of other determining postulates, still referring to profits and describing a form of competition, but substantially different from the one which has just been analyzed. Each competitor may by slightly changing the price—say by under-selling another—try to obtain whatever portion of the trade he can handle with

the maximum return or profit. This attitude with reference to a steady situation may be described as "cut-throat" or strict competition, whereas the same attitude with reference to a temporary one produces the familiar "sale." The producer regards the price as fixed—say, just less than the market price—and produces the amount which would give him the maximum profit at that price. In order to determine the steady state for this sort of competition we may state the postulate as follows.

Each competitor regards the price as fixed and tries to make his profit a maximum.

In this case we have the equations:

$$\left(\frac{\partial \pi_i}{\partial u_i}\right)_{p \text{ constant}} = 0, \quad i = 1, 2, \tag{10}$$

whence, by differentiating the formulae for π_i, keeping in mind that p is to be treated as a constant,

$$0 = p - q'(u_1) = p - q'(u_2)$$

or

$$p = 2Au_1 + B = 2Au_2 + B \tag{11}$$

With reference to (2) then we calculate the solutions:

$$\begin{aligned} p &= \frac{Ab + B}{1 - Aa}, \quad u_1 = u_2 = \frac{b + Ba}{2 - 2Aa}, \\ u &= u_1 + u_2 = \frac{b + Ba}{1 - Aa}. \end{aligned} \tag{12}$$

We are thus led back to precisely the situation of Sec. **7**, in which (11) determines the individual offers u_1 and u_2 in terms of p, and (12) results from equating the offer and the demand. It is interesting to notice that here, with the case of two producers, each manufacturer separately produces as much as the total production under the monopoly postulate, given by equation (6) of Chapter I. This particular result depends of course on the particular cost and demand functions which we have assumed.

EXERCISE.—If the coefficients of the two cost functions are A_1,B_1,C_1; A_2,B_2,C_2, respectively, instead of A, B, C, show that

$$\begin{aligned} p &= \frac{2A_1A_2b + A_2B_1 + A_1B_2}{A_1 + A_2 - 2A_1A_2a} \\ u &= u_1 + u_2 = \frac{(A_2B_1 + A_1B_2)a + (A_1 + A_2)b}{A_1 + A_2 - 2A_1A_2a} \end{aligned} \tag{13}$$

and that the same substitution adopted in the example on coöperation reduces (13) to (12).

19. Phenomena with n Producers and Relative Values.—For purposes of comparison we may use a superscript m to refer to monopoly, c for cooperation, a for the Cournot form of competition of Sec. **17** and b for the second form of competition, just described. The reader will easily verify that if there are n producers instead of two, the analysis relative to cooperation and competition can be carried out as before and the following formulae will be obtained:

$$p^{(c)} = \frac{nb - 2Aab - nBa}{-a(2n - 2Aa)}, \quad u_i^{(c)} = \frac{b + Ba}{2n - 2Aa} \tag{14}$$

$$p^{(a)} = \frac{b - 2Aab - nBa}{-a(n + 1 - 2Aa)}, \quad u_i^{(a)} = \frac{b + Ba}{n + 1 - 2Aa} \tag{15}$$

$$p^{(b)} = \frac{2Ab + nB}{n - 2Aa}, \quad u_i^{(b)} = \frac{b + Ba}{n - 2Aa} \tag{16}$$

If we let n become large, the terms that involve n will overshadow those that do not. Thus the approximate value of $2n - 2Aa$ will be $2n$. For large values of n we may therefore write

$$p^{(a)} = p^{(b)} = B, u_i^{(a)} = u_i^{(b)} = \frac{b+Ba}{n}, u^{(a)} = u^{(b)} = b + Ba \tag{17}$$

$$p^{(c)} = \frac{b - Ba}{-2a}, u_i^{(c)} = \frac{b + Ba}{2n}, u^{(c)} = \frac{b + Ba}{2}, \tag{17.1}$$

so that the total production in the case of cooperation would be half that for either form of competition, these becoming sensibly identical for large n.

There are one or two other special cases that may be noted in passing. If we put $n = 1$, the prices $p^{(a)}$, $p^{(c)}$ become identical with the $p^{(m)}$, for monopoly. This is not the case however with the $p^{(b)}$ which reduces to

$$p^{(b)} = \frac{2Ab + B}{1 - 2Aa}, \text{ with } u^{(b)} = \frac{b + Ba}{1 - 2Aa} \tag{18}$$

Another situation to note is where the demand y is independent of the price *i.e.*, where $a = 0$. In this case

$$p^{(m)} = p^{(c)} = p^{(a)} = \infty \tag{19}$$

or, in other words, the prices are pushed up beyond the region in which the hypothesis of demand independent of price remains valid, not only for monopoly and cooperation, but also for

competition as described by Cournot. On the other hand, for $p^{(b)}$ we get the value

$$p^{(b)} = \frac{2Ab}{n} + B, \text{ with } u_i^{(b)} = \frac{b}{n} \tag{19.1}$$

The two kinds of competition represent therefore quite distinct situations.

Some essential differences are brought out also if we consider an industry where aside from overhead cost most of the cost is labor—a situation which we may characterize by writing $A = 0$. We obtain

$$p^{(c)} = p^{(m)} = \frac{B}{2} - \frac{b}{2a}, \quad u^{(c)} = u^{(m)} = \frac{b + Ba}{2}$$

$$p^{(a)} = \frac{n}{n+1}B - \frac{b}{(n+1)a}, \quad u^{(a)} = \frac{n}{n+1}(b + Ba),$$

$$p^{(b)} = B, \quad u^{(b)} = b + Ba, \tag{20}$$

so that the price and total production for Cournot competition are the only ones that depend on the number of producers engaged.

The values in (20) which refer to strict competition are given for the sake of completeness, but, as may be seen from the graph of $q(u_i)$, they do not correspond to a practical interpretation of the problem.

In view of these results, it is opportune to prove that in general, that is, *for an arbitrary finite value of* $n > 1$ *and for* $A > 0$, $a < 0$, *the prices and amounts produced stand in a definite sequence in order of magnitude, namely*

$$p^{(m)} > p^{(c)} > p^{(a)} > p^{(b)} \tag{21}$$

$$u^{(m)} < u^{(c)} < u^{(a)} < u^{(b)}$$

It is enough to prove one of these sequences of inequalities, since the other follows from it by means of the relation $u = \Sigma u_i = ap + b$, $a < 0$.

In fact from (14), (15), (16), we see directly that since

$$2n - 2Aa > n + 1 - 2Aa > n - 2Aa$$

we have

$$u_i^{(c)} < u_i^{(a)} < u_i^{(b)}$$

and therefore

$$u^{(c)} < u^{(a)} < u^{(b)}.$$

But also

$$u_i^{(c)} = nu_i^{(c)} = \frac{b + Ba}{2 - 2(Aa/n)}$$

which is greater than $u^{(m)} = (b + Ba)/(2 - 2Aa)$ since $-Aa/n$ is positive and less than $-Aa$. Hence $u^{(m)} < u^{(c)}$, and the sequence is complete.

As we have seen by means of the exceptional cases previously discussed the restrictions mentioned in the hypothesis are necessary for the truth of the theorem just given. Those special cases will however also be covered if we replace the signs $<$ by $\leq$ and the signs $>$ by $\geq$.

20. Profit and Loss.—If the price falls to a certain value, any producer will fail to make a profit, and if it falls still lower, since $C > 0$, the producer will lose at a definite rate. It remains to find this critical price.

Consider the u, q plane in which the graph of the cost function is represented (Fig. 10). A line $s = pu_i$, in which p is an arbitrary constant, cuts the cost curve $q = Au_i^2 + Bu_i + C$ in two points, real and distinct, coincident or imaginary. It is only in the first of these three cases that there is a portion of the line above the curve, that is, since s gives the selling value of the amount u_i, where it is possible to choose a value of u for which the selling value will exceed the cost. The critical value of p will therefore be the one which makes the price line cut the cost curve in two coincident points (the reader may show that these two points will coincide at the point of the curve where the average and the marginal unit costs are identical[1].) To determine the points of intersection we have $s = q(u_i)$ or

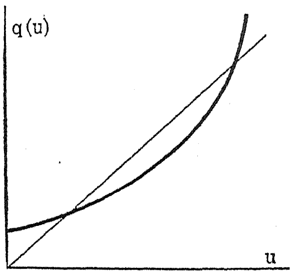

FIG. 10.

$$0 = Au_i^2 + (B - p)u_i + C,$$

and they will therefore be coincident when the right-hand member is a perfect square, that is when

$$(B - p)^2 = 4AC$$

This yields the value of p

$$p = B + 2\sqrt{AC},$$

the plus sign being taken with the radical, since the other value of p would correspond to a negative value of u, as the shape of the cost curve makes evident.

[1] Compare Exercise 1, Sec. 8.

The lowest of the prices is $p^{(b)}$. The condition $p^{(b)} \geq p$ is equivalent to the condition

$$nB + 2Ab \geq (n - 2Aa)(B + 2\sqrt{AC}),$$

from which we can deduce the condition

$$n < \sqrt{\frac{A}{C}}(b + Ba + 2a\sqrt{AC}). \tag{21.1}$$

This is a necessary condition for profit under the second form of competition. It is also a sufficient condition, for the value of u is determined by the equation $p = q'(u_i)$, and the value of u_i which corresponds to p by this equation (which says that the slope of the tangent to the curve at the point is p) must lie between the two values which give the intersections of the line $s = pu_i$ with the cost curve, when these are real.

The equation (21.1) gives therefore a definite finite limit on the number of competitors who may safely engage in a given production. It is not sufficient to say, as some economists do, that the profit for any competitor tends to approach zero as the number becomes indefinitely large, but rather that the producers must actually produce at a loss if their number exceeds the value n given by (21.1).

The same situation may be brought out in another way. It is conceivable that the A, B, C, particularly the C, depend on n, the number of equal competitors, inasmuch as the overhead cost in a business is usually small if the business is small. We may regard as an interesting and perhaps practical assumption that A, B, are independent of n, but that C is inversely proportional to n, *viz.*—$C = K/n$ where K is constant, not changing with n. With this assumption a new inequality is easily found. In fact, from the equality which precedes (21.1),

$$nB + 2Ab \geq (n - 2Aa)\left(B + \frac{2\sqrt{AK}}{\sqrt{n}}\right)$$

whence

$$Ab + ABa \geq \sqrt{n}\sqrt{AK} - \frac{2Aa\sqrt{AK}}{\sqrt{n}}$$

or

$$\sqrt{\frac{A}{K}}(b + Ba) \geq \sqrt{n} - \frac{2Aa}{\sqrt{n}},$$

but since

$-2Aa/\sqrt{n}$ is positive we must have all the more

$$\sqrt{\frac{A}{K}}(b + Ba) \geq \sqrt{n}, \tag{22}$$

and this furnishes the inequality which limits n. For competition of kind (b), n must at least satisfy this inequality if there is to be any profit.

Further light on the question is thrown by considering n large, as economists do in many theories. This can be done by making use of the formulae (17). Consider first the two kinds of competition (a), (b) which have now become identical. For the profit π_i of an individual producer we have

$$\pi_i = Bu_i - Au_i^2 - Bu_i - C$$

or

$$\pi_i = -Au_i^2 - C \tag{23}$$

since $p = B$. But this is essentially negative. In other words, n has already got beyond the value where profit is possible if the approximations used in (17) are to be made.

With cooperation however the case is different. There we have for $\pi_i = pu_i - q(u_i)$ the value

$$\pi_i = \left(\frac{b - Ba}{-2a}\right)\left(\frac{b + Ba}{2n}\right) - A\frac{b^2 + 2Bab + B^2a^2}{4n^2} - B\frac{(b + Ba)}{2n} - \frac{K}{n},$$

so that we have, simplifying and grouping the terms,

$$-4an^2\pi_i = n(b^2 + B^2a^2 + 2Bab + 4aK) + Aa(b + Ba)^2,$$

so that

$$\pi_i = \frac{(b + Ba)^2 + 4aK}{-4an} - \frac{A(b + Ba)^2}{4n^2}$$

and for large values of n,

$$\pi_i = \frac{(b + Ba)^2 + 4aK}{-4an}, \tag{24}$$

This number is positive if

$$K < \frac{(b + Ba)^2}{-4a}, \tag{24.1}$$

no matter what the value of n; but the magnitude of π_i (for large values of n) is inversely proportional to n. Hence in this case

also there is an effective limit on n. A significant result is that for large values of n the total profit $= \Sigma\pi_i$ is independent of n, as follows from (24).

On the whole the situation just described has a strong resemblance to the phenomenon of trade unionism, where the members of the union associate in such a way as to get the maximum profit from their combined services.

21. Modifications in the Cost Curve.—Up to this point we have been considering situations in which time was not a determining factor. They have been steady states, or states of equilibrium. The rates of production, prices, costs, etc., have been connected directly by equations among themselves without any need of introducing the time as a variable. But we can conceive of the desirability of investigating situations which are not in equilibrium, in order to see where they tend, and how rapidly. This suggests the ~~dymanic~~ namic aspect rather than the static aspect of economic problems, and is the subject of a later part of our course.

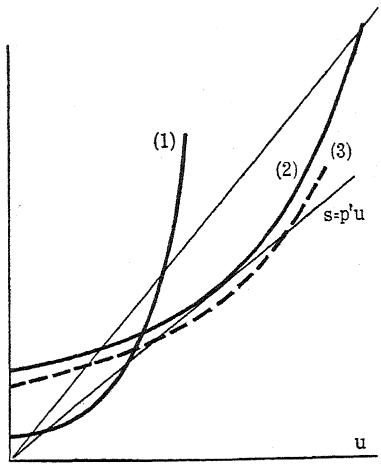

FIG. 11.

Consider for instance a situation which has been in equilibrium, but where the cost function has changed. What happens in the system? At this point we may take up a special case graphically. Rather than consider the total variety of cost curves among producers, let us consider a type of modification which is characteristic of our problem.

Suppose that the situation is one of equilibrium with competition of the type (b). Some of the producers may wish to make a larger profit by increasing their overhead expenses, decreasing their other expenses relatively, and producing larger amounts. In the diagram, the curves (1) and (2) represent an individual cost curve before and after such a change, the amount which would be produced in each case being that which would make the vertical distance between the line $s = pu$ (which represents the selling value) and the cost curve a maximum. As we have

seen, this is the position in which the tangent to the cost curve is parallel to the price line,—*i.e.*, $p = q'(u)$. The transition contemplated is from (1) to (2), in Fig. 11.

In (*b*), as we have seen, an offer curve is defined, so that offer is given as a function of price. The transition from (1) to (2), however, changes this function, and as is evident from the remark just made, will increase the offer at the price p. Since $a < 0$, this will decrease the price, and if the price goes below a certain value p', indicated on the diagram by the line $s = p'u$ there will no longer be any profit in the expanded business. The change from (1) to (2) is unfortunately usually not a reversible process. And the usual modification of the cost curve which is now enforced is from (2) to (3), by omitting to pay dividends, and by such other changes in the value of C as may ultimately provoke the bond holders to demand a receivership. Thus apart from any phenomenon connected with the rate of interest,[1] bankruptcy may be regarded as a normal event in a system of competition.

22. General Exercises.

1. It will be noticed that few of the formulae involve C and thus most of them will not depend on the overhead cost. We have taken up to this point the quadratic function as a representation of the cost curve throughout its whole extent. If however the quadratic function gives the cost curve approximately only as in Fig. 12 through a portion of the curve, we can still say $A > 0$, if we assume that the marginal unit cost increases with u, but not any longer $B > 0$, $C > 0$, as this same figure indicates. Consider then the results of this chapter and see how many of them remain applicable without assuming $B > 0$, $C > 0$, but assuming of course that all the values obtained remain within the portion of the curve where the approximation is valid.

2. Consider the situation in which there are two competitors, one of them operating in accordance with a postulate of type (*a*), the other as if the postulate were of type (*b*). Consider the case of three producers, one a competitor according to type (*b*), the others endeavoring to cooperate.

3. Analyze the following paradox. In cooperation the total profit is maximized. In competition (*a*) each producer makes his own profit a maximum. But the total profit is merely the sum of the individual profits, and therefore will also be a maximum when the individual profits are at a maximum. Therefore the two situations are the same, in contradiction with the results which we have obtained.

4. Show, by calculating $\partial^2\pi_1/\partial u_1^2$ and $\partial^2\pi_2/\partial u_2^2$, that if $A < 0$ the result expressed in (12) no longer yields a maximum profit. This means that an equilibrium situation is impossible for competition of type (*b*) if the marginal unit cost $2Au_i + B$ is a decreasing instead of an increasing function of u_i.

[1] See Chapter VIII.

5. If the marginal unit cost $q'(u_i)$ is a decreasing function of u_i, the situation is said to be one of *decreasing costs*. Show that in this situation, whatever the demand and cost functions may be, there is no possible equilibrium under competition of type (b). Take $y = \varphi(p)$, $q_i = q(u_i)$, without assuming that $\varphi(p)$ is linear and $q(u_i)$ quadratic.

Investigate the same problem graphically.

6. Assuming $A < 0$, obtain restrictions on the A, a etc., so that there will still be positions of maximum profits for monopoly, cooperation and competition of type (a). What order relations can be stated among the several prices of equilibrium?

7. Perform a similar analysis, assuming that $y = \varphi(p)$, $q_i = q(u_i)$ without assuming that $\varphi(p)$ is linear and $q(u)$ quadratic. Distinguish between the cases of increasing and decreasing costs.

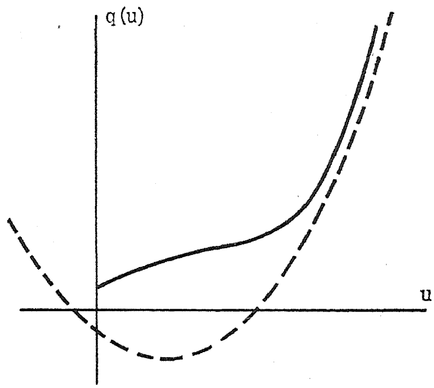

FIG. 12.

8. It may happen that a monopoly, especially when government operated, does not seek a maximum profit, but proceeds according to some other criterion.[1] Consider, for instance, the price and production if $q = Au^2 + Bu + C$ and equilibrium is obtained by producing enough to make the cost equal to the selling value, the demand being $y = ap + b$. What is the relation of the price so obtained to that of the earlier monopoly problem?

9. Draw graphs on the same axes of $q(u)$, $s = pu = \frac{u - b}{a} u$ and $v = \text{arc tan } p = \text{arc tan } \frac{u - b}{a}$, and thus compare the equilibrium productions for monopoly in the usual form, the situation of Sec. **6** and that of the previous exercise.

10. Consider the equilibrium price and production if there are n producers, and for each producer the cost is equal to the selling value, the demand being $y = \Sigma u_i = ap + b$, and the cost functions being all identical and quadratic in form. Compare with competition of type (b).

11. Obtain the results of exercises 10, 11, 12 of Sec. **8** by substituting new values of the coefficients a, b, A, B, C in the formulae for the problem of monopoly.

[1] AMOROSO, "Lezioni di Economia Matematica," p. 272, Bologna, 1921.

CHAPTER IV

CASES OF VARIABLE PRICE

23. A New Type of Offer and Demand.—In the actual situation of society we know that prices change from time to time, and, in fact, rarely remain constant, so that they cannot in general be solutions of equations as simple as those already given. Accordingly, we shall not have an adequate picture of economics until we invent likely hypotheses which will make the price vary with time, and enable us to determine it for various times, that is, as a function of the time. In other words we desire to be in a position to be able to plot the price as a graph against the time.

One possibility of course is that we should replace the constants A, B, C, a, b, by variables $A(t)$, $B(t)$, etc., functions of the time. Given however the difficulty of formulating the behaviour of such functions, we should look to see if there is any other way that we can get variable prices. Possibilities in this direction have already been indicated in Sec. **3**.

We may say, for instance, that the prices already obtained are to be regarded as equilibrium prices, but that prices, instead of being in equilibrium, are usually functions of the time which have those equilibrium prices as limiting values. Whether the price is going up or down is itself an important factor in the demand for the quantity. In actual cases the demand is often not merely a function of the price alone but is stimulated or depressed by the mere fact that the price is rising or falling. We know that business is usually good when prices are rising and usually not so good when prices are falling; the number of shoes that will be bought at three dollars a pair will be greater if it is known that the price is increasing at the rate of fifty cents a week than if the price is supposed to be decreasing at the rate of fifty cents a week.

A good approximation for such a situation is to assume a law of demand as follows

$$y = ap + b + h\frac{dp}{dt}, \tag{1}$$

where $a < 0$, $b > 0$ and where, to exemplify the situation just

described, we have also $h > 0$, the quantity dp/dt being the rate of increase of price.

In order to get a situation that can be easily handled, let us assume that there is a total offer function, more or less of the same kind:

$$u = u_1 + u_2 + \cdots + u_n = \alpha p + \beta + \gamma \frac{dp}{dt}. \tag{2}$$

Here we shall naturally take $\alpha > 0$, $\gamma > 0$, and since when $dp/dt = 0$ there will be no offer unless p is at least as great as some minimum positive value, we shall take $\beta < 0$, although the latter restriction has little effect on the results.

If finally we assume that the offer is equal to the demand,

$$u = y$$

we have to determine p, as a function $p(t)$ of t, the equation

$$ap + b + h\frac{dp}{dt} = \alpha p + \beta + \gamma \frac{dp}{dt},$$

or

$$\frac{dp}{dt} + \frac{\alpha - a}{\gamma - h} p = -\frac{\beta - b}{\gamma - h}. \tag{3}$$

Conversely, any function $p(t)$ which satisfies (3) will also satisfy our assumptions, if (1) and (2) are used to define y and u.

In particular, since the derivative of a constant is zero, this equation has as a possible solution the constant

$$p_1 = -\frac{\beta - b}{\alpha - a},$$

a positive quantity which accords with the value described in Chapter I, Sec. 7.

Let now p be any function of t which satisfies (3), and $z = p - p_1$. Then z satisfies the simpler equation

$$\frac{dz}{dt} + \frac{\alpha - a}{\gamma - h} z = 0 \tag{4}$$

as we see by direct substitution, and comparison with (3). But this equation is the same as

$$\frac{dt}{dz} = -\frac{\gamma - h}{(\alpha - a)z}$$

which has the solution

$$t = -\frac{\gamma - h}{\alpha - a} \log z + \text{const.}$$

This relation may be written in the form

$$z = Ce^{-\frac{\alpha - a}{\gamma - h}t}.$$

In fact this formula may be verified by substituting directly in (4), although the treatment just used has the advantage of showing that the formula contains all the solutions z, C being an arbitrary constant.

We obtain then finally for the general solution of (3) the result

$$p(t) = p_1 + Ce^{-\frac{\alpha - a}{\gamma - h}t}$$

In this formula, knowing the value of C determines the price completely as a function of t. If for instance, we know that $p(t)$ has some given value p_0, when $t = 0$, we find by substitution that

$$p_0 = p_1 + C, \quad C = p_0 - p_1,$$

so that

$$p(t) = p_1 + (p_0 - p_1)e^{-\frac{\alpha - a}{\gamma - h}t} \tag{5}$$

What is interesting is to notice what happens with the lapse of time. Since

$$\lim_{x = +\infty} e^x = \infty, \lim_{x = -\infty} e^x = 0,$$

we have the two cases

$$\text{if } \frac{\alpha - a}{\gamma - h} > 0, \quad \lim_{t = \infty} p(t) = p_1 \tag{5.1}$$

$$\text{if } \frac{\alpha - a}{\gamma - h} < 0, \quad \lim_{t = \infty} p(t) = +\infty \quad \text{when } p_0 - p_1 > 0, \tag{5.2}$$
$$= -\infty \quad \text{when } p_0 - p_1 < 0.$$

The practical meaning of the case (5.2) is that the price flies off beyond the region in which the hypotheses (1), (2) with $u = y$ are tenable. The condition for stability in the market is that we should have the case (5.1), and since we assume that $a < 0, \alpha > 0$, this condition reduces merely to the inequality

$$\gamma > h \tag{6}$$

The constant $p = p_1$, is a solution, of course, in both cases, but in (5.2) if p has at any time a value even slightly different from p_1 it will diverge from that value more and more, while in (5.1) it will tend to return to the value p_1.

If we think of the coefficients h, γ as indices of sensitivity of demand and offer to change of price, we may say roughly that

if the sensitivity of demand is the greater the situation is unstable, while if the sensitivity of offer is the greater the situation is stable, in that the price tends towards a definite limit.

24. Graphical Study of Price as a Function of Time.—The equation (3) was simple in the sense that we could find the

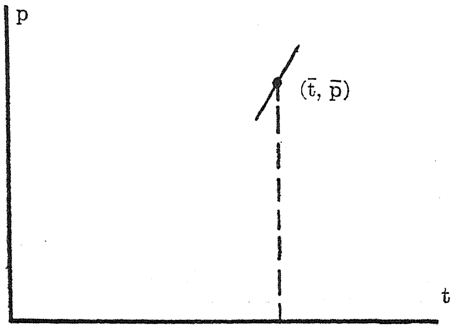

FIG. 13.

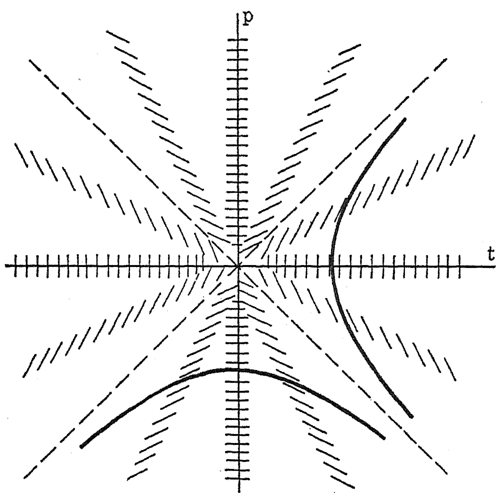

FIG. 14.

explicit expression (5) for the solutions of it. But we cannot hope that all of our relations will be solvable in this ready manner. In more difficult cases, however, it is still possible to treat the equations graphically and find out readily the qualitative nature of the result.

Equation (3) is a special case of the more general relation

$$\frac{dp}{dt} = f(t, p) \tag{7}$$

where $f(t, p)$ is some given function of p and t. This relation says that for given values of t and p the rate of change of p with respect to t is determined, or that the graph of p as a function of t is such that if for some value $t = \bar{t}$, p happens to have the value $\bar{p}$, then $f(\bar{t}, \bar{p})$ will give the value of the slope of the tangent to the graph. We may accordingly plot these slopes at all the points (t, p) in the plane and then draw curves which have only these lines as tangent lines. Thus if

$$\frac{dp}{dt} = \frac{t}{p}$$

we have the accompanying graph (Fig. 14).

In fact, we have the following table of values

t	p	dp/dt
0	0	indeterminate
1	0	∞
0	1	0
0	2	0
1	1	1
2	2	1
-1	-1	1
All points such that $p = t$		1
All points such that $p = 2t$		$\frac{1}{2}$
All points such that $p = -t$		-1
All points such that $p = -2t$		$-\frac{1}{2}$
All points such that $p = \frac{1}{2}t$		2
All points such that $p = 0$		∞
All points such that $t = 0$		0
Etc.		

When these directions are plotted thickly over the plane there is no difficulty in drawing curves which have everywhere the proper direction.

For the equation (3) the lineal elements, as we may call them, are easily plotted. In fact, since t does not enter into the second member all the points in the plane which have the same value of p will have parallel lineal elements. The equation may be written in the form

$$\frac{dp}{dt} = lp + m$$

where $l < 0$ for case (5.1) and $l > 0$ for case (5.2), and m and l have opposite algebraic signs.

Particular values may be given to the constants l and m for the plotting.

25. Another Demand Law.—The demand law just discussed seems to be a likely practical hypothesis if the changes of price

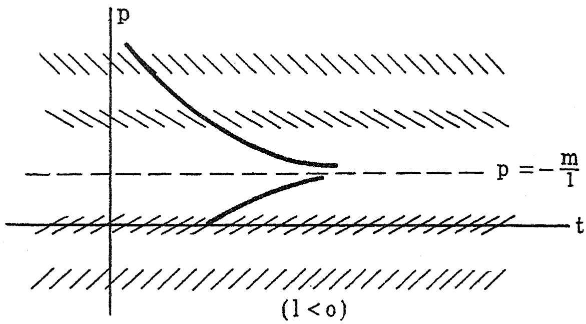

FIG. 15.

are not too abrupt, *i.e.*, if dp/dt is small in numerical value. But since it is desirable sometimes to consider very rapid changes of price, and even to allow dp/dt to become infinite, the law is obviously not sufficiently precise, even in a qualitative sense, to

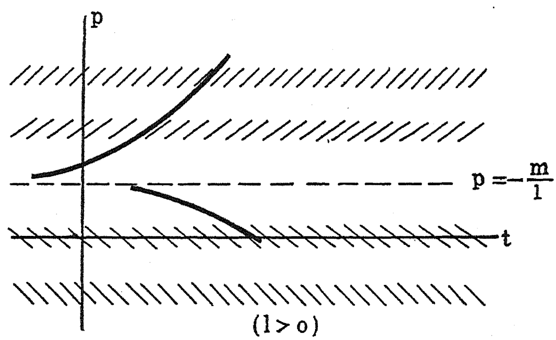

FIG. 16.

cover all the interesting situations. We must, for instance, take refuge in the statement that if

$$\frac{\alpha - a}{\gamma - h} < 0$$

the value of p will then progress beyond a point where the hypothesis is even qualitatively applicable; otherwise p would pass all possible values.

We might try to avoid the difficulty by replacing the term $h\dfrac{dp}{dt}$ with one which behaves like it for small values of $\dfrac{dp}{dt}$ but which never becomes infinite, so that y cannot be infinite without p being infinite. Let us write then:

$$
\begin{aligned}
y &= ap + b + h \operatorname{arc\,tan} \frac{dp}{dt}. \\
u &= \alpha p + \beta + \gamma \operatorname{arc\,tan} \frac{dp}{dt} \\
u &= y
\end{aligned}
\tag{8}
$$

In these equations, in order that the terms involving h arctan dp/dt be similar to hdp/dt, for small values of the derivative, we

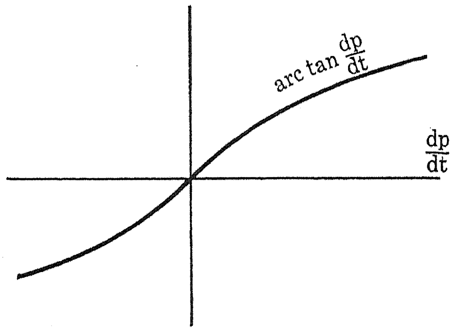

FIG. 17.

must choose as the graph of arctan dp/dt the one which passes through the origin; in other words, we must consider only values of arctan dp/dt between $-\pi/2$ and $+\pi/2$.

The price p will then satisfy the equation:

$$(\alpha - a)p + \beta - b + (\gamma - h) \operatorname{arc\,tan} \frac{dp}{dt} = 0 \tag{9}$$

and on account of the restriction just mentioned, this equation cannot be satisfied unless p is subject to the restriction:

$$\left| \frac{(\alpha - a)p + (\beta - p)}{\gamma - h} \right| \leq \frac{\pi}{2}. \tag{10}$$

If $\gamma = h$, the solution is

$$p_1 = -\frac{\beta - b}{\alpha - a}$$

as before, and if $\alpha = a$, we have

$$\frac{dp}{dt} = -\tan\frac{\beta - b}{\gamma - h}, \quad p = p_0 - t\tan\frac{\beta - b}{\gamma - h}$$

Otherwise, which is the more general case, we have from (9),

$$\begin{aligned} \text{arc tan}\,\frac{dp}{dt} &= -\frac{\alpha - a}{\gamma - h}p - \frac{\beta - b}{\gamma - h} \\ \frac{dp}{dt} &= \tan\left[-\frac{\alpha - a}{\gamma - h}p - \frac{\beta - b}{\gamma - h}\right] \\ \frac{dt}{dp} &= -\cot\left[\frac{(\alpha - a)p + \beta - b}{\gamma - h}\right] \end{aligned} \tag{11}$$

If the initial value of p is such as to make the [ ] positive and less than $\pi/2$, we have

$$t = -\frac{\gamma - h}{\alpha - a}\log\sin\left[\frac{(\alpha - a)p + \beta - b}{\gamma - h}\right] + C',$$

On the other hand, if the initial value of p is such as to make the same bracket negative, we have

$$t = -\frac{\gamma - h}{\alpha - a}\log\sin\left[-\frac{(\alpha - a)p + \beta - b}{\gamma - h}\right] + C''$$

where C' and C'' are constants.

These equations may be written respectively in the forms:

$$K'e^{-\frac{\alpha - a}{\gamma - h}t} = \sin\frac{(\alpha - a)p + \beta - b}{\gamma - h}$$

and

$$K''e^{-\frac{\alpha - a}{\gamma - h}t} = -\sin\frac{(\alpha - a)p + \beta - b}{\gamma - h},$$

where K' and K'' are formed from C' and C'' by the formula $K = e^{-\frac{\alpha - a}{\gamma - h}C}$. But if we let p_0 be the value of p when $t = 0$, both of these equations take the same form

$$\left\{\sin\frac{(\alpha - a)p_0 + (\beta - b)}{\gamma - h}\right\}e^{-\frac{\alpha - a}{\gamma - h}t} = \sin\frac{(\alpha - a)p + (\beta - b)}{\gamma - h} \tag{12}$$

If we assume that the constants α, b, h, γ are all ≥ 0 and that $a < 0$, as usual, the character of the relation between p and t will depend on the sign of $\gamma - h$. For if $\gamma > h$,

$$\lim_{t=\infty} e^{-\frac{\alpha - a}{\gamma - h}t} = 0,$$

and, regardless of the value of p_0 within the range given by (10), p tends towards the equilibrium value

$$\lim_{t=\infty} p = p_1 = -\frac{\beta - b}{\alpha - a}$$

On the other hand, if $\gamma < h$,

$$\lim_{t=\infty} e^{-\frac{\alpha - a}{\gamma - h}t} = +\infty$$

Hence in this case there will be a definite value of t for which the left-hand member of (12) will become equal to 1 or -1, and after that the equation cannot any longer be satisfied, since the sine of a real quantity cannot be greater than 1 in numerical value.

The two values of p at which the equations become meaningless if $\gamma < h$ are given by

$$\frac{(\alpha - a)p + (\beta - b)}{\gamma - h} = \pm\frac{\pi}{2}$$

or

$$p = \frac{\pm\pi(\gamma - h) - 2(\beta - h)}{2(\alpha - a)} \tag{13}$$

We see by comparison with (11) that for these values we should have $dp/dt = \infty$.

The two cases are then as follows:

(*a*) Stability; with $\gamma > h$, where no matter what price we start from in the allowable range, we tend asymptotically towards the equilibrium price $p_1 = -\frac{(\beta - b)}{(\alpha - a)}$.

(*b*) Instability; with $\gamma < h$, where no matter what price we start from, other than $p_0 = p_1$, we come after a certain interval of time to one of the prices given by (13), after which the equations (8) can no longer be satisfied by real values of p. The price reaches a value beyond which the hypotheses of the problem can no longer hold, and new ones must be substituted. For example, at this point h may take a new value $< \gamma$, or the b and β may take new values.

The directions dp/dt which satisfy the equation are exemplified by the following two diagrams. The curves which give p as a function of t are those which at every point have the plotted direction, that is, which satisfy (9).

These lineal elements may be easily plotted by noticing that equation (11) states that the angle which the lineal element makes with the horizontal is given by the quantity

$$\mu = -\frac{(\alpha - a)p + (\beta - b)}{\gamma - h}$$

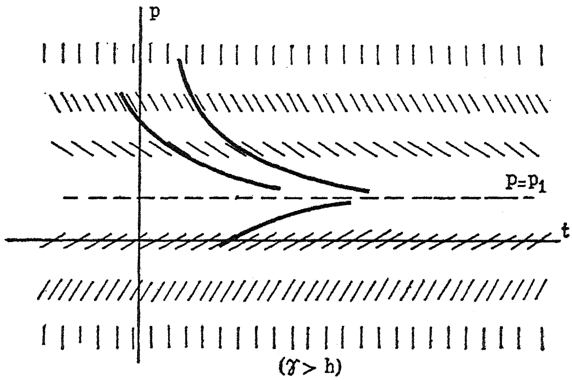

FIG. 18.

since dp/dt is the tangent of that angle. Hence these elements are horizontal when $\mu = 0$, and vertical when $\mu = \pm\pi/2$, and the angles they make with the horizontal are proportional to the distance of the point from the line where $\mu = 0$, that is, where $p = p_1$.

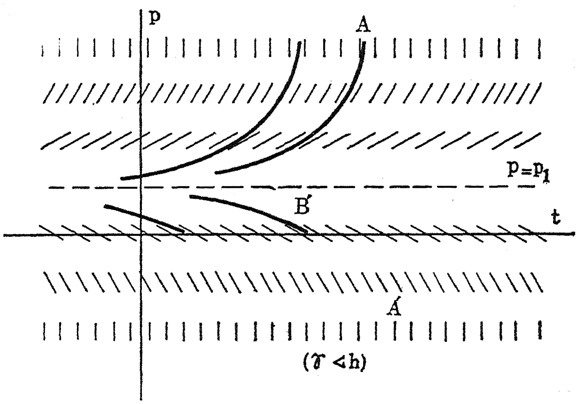

FIG. 19.

The values of p for which $\mu = \pm\pi/2$ are given by (13); they are the values, when $\gamma < h$, at which the hypotheses cease to be tenable. In fact, in this case, when the curve reaches such a point it ceases to go further to the right and therefore cannot

provide for increasing values of t. This situation is illustrated by such points as A and A' in the second figure. Of course the point A', as the figure is drawn, is itself inaccessible, since negative values of p are not admissable, and the curve would have to stop at B'. But there is nothing to prevent the whole figure from being above the t-axis, as is seen by (9), if the constants are properly chosen.

26. An Integral Demand Law.—Another situation involving the change of prices occurs if we assume that the demand for a commodity depends not only on the present price but on the history of the price, depending less and less on past prices as the time is more and more remote. Thus we might write

$$y(t) = ap(t) + b + a_1p(t - r) + a_2p(t - 2r) + \cdots + a_kp(t-kr) \quad (14)$$

where $a_1, a_2, \ldots a_k$ form a sequence of constants decreasing in numerical value, and r is a suitable interval of time, say a week or a month; in this way the demand would depend on the price at the time and at times one week, two weeks, etc., before. We might even make $y(t)$ depend on the price at all times before the time t.

An instance of the last sort of dependence is given by the assumption

$$y(t) = ap(t) + b + h\int_{-\infty}^{t} e^{-m(t-\tau)}p(\tau)d\tau \quad (15)$$

where $a < 0$, $b > 0$, $m > 0$; and h, as it turns out, must be positive. In this formula $e^{-m(t-\tau)}$ is negligeable if $m(t - \tau)$ is at all large, so that the main effect of previous prices is that only of prices fairly recent. The equation (15) may be regarded as a sort of limiting case of one like (14).

Let us assume again an offer, in the simplest form,

$$u(t) = \alpha p(t) + \beta \quad (16)$$

with $\alpha > 0$, $\beta < 0$, and take

$$y = u.$$

The equation to determine p is obtained by subtracting (15) from (16) and may be written in the form

$$(\alpha - a)p(t) = -(\beta - b) + h\int_{-\infty}^{t} e^{-m(t-\tau)}p(\tau)d\tau \quad (17)$$

It is beyond the scope of the present manual to give solutions of of integral equations like (17) in general. But the particular equation (17) may be solved by differentiation. In fact, from (17),

$$(\alpha - a)\frac{dp}{dt} = hp(t) - hm\int_{-\infty}^{t} e^{-m(t-\tau)}p(\tau)d\tau,$$

and by comparing this equation again with (17)

$$(\alpha - a)\frac{dp}{dt} = hp(t) - m(\alpha - a)p(t) - m(\beta - b)$$

or

$$\frac{dp}{dt} = \left(\frac{h}{\alpha - a} - m\right)p(t) - m\frac{\beta - b}{\alpha - a} \tag{18}$$

This last equation is similar to (3) and may be solved in the same way. We may verify at once however by substitution that it is satisfied by the expression

$$p(t) = \frac{m(\beta - b)}{h - m(\alpha - a)} + Ce^{\left(\frac{h}{\alpha - a} - m\right)t} \tag{19}$$

where C is a constant, as yet undetermined. It would seem that the C might be determined by substitution into (17); but this turns out not to be the case.

The substitution yields

$$\frac{m(\beta - b)(\alpha - a)}{h - m(\alpha - a)} + C(\alpha - a)e^{\frac{h - m(\alpha - a)}{\alpha - a}t} = -(\beta - b)$$
$$+ \frac{hm(\beta - b)}{h - m(\alpha - a)}\int_{-\infty}^{t} e^{-m(t-\tau)}d\tau + hC\int_{-\infty}^{t} e^{-m(t-\tau)}e^{\frac{h - m(\alpha - a)}{\alpha - a}\tau}d\tau.$$

The right-hand member of this equation is

$$-(\beta - b) + \frac{h(\beta - b)}{h - m(\alpha - a)} + hCe^{-mt}\int_{-\infty}^{t} e^{\frac{h\tau}{\alpha - a}}d\tau,$$

and the integral is not convergent unless h is positive. With this restriction the equation written above becomes the following:

$$C(\alpha - a)e^{\frac{h - m(\alpha - a)}{\alpha - a}t} = C(\alpha - a)e^{-mt}e^{\frac{h}{\alpha - a}t},$$

which is an identity. Hence (19) is a solution of (17) whatever the value of C provided $h > 0$. The C may of course be given in terms of some assumed initial value p_0 of $p(t)$, say for $t = 0$.

Here again there are the two situations of stability and instability. If $h > m(\alpha - a)$ the situation is unstable since $\lim p(t) = \pm\infty$. On the other hand, if h, still positive, is $< m(\alpha - a)$, the quantity $\frac{h - m(\alpha - a)}{\alpha - a}$ will be < 0 and

$$\lim_{t=\infty} p(t) = \frac{m(\beta - b)}{h - m(\alpha - a)}$$

This limiting value is itself a constant solution of (17), obtained by putting $C = 0$ in (19), but it is not the solution we should get by writing $h = 0$ in (17), as the limiting value was in the cases of equations (3) and (9).

27. General Exercises.

1. Consider the situation where there is a demand

$$y = ap + b$$

and a total offer

$$u = \alpha p + \beta,$$

where y is not necessarily equal to u, but where the rate of change of the quantity $u - y$ is proportional to that quantity itself. Should the constant of proportionality be assumed to be positive or negative? What is its dimension formula?

Obtain p as a function of the time and compare the result with some of the situations already discussed in this chapter.

Find y and u as functions of the time. What happens when t becomes infinite?

2. Consider a demand law of the form (14) where all the coefficients a_i are zero after $a_1 = h$; viz.,

$$y(t) = ap(t) + b + hp(t - r).$$

Assume that there is an offer equal to the demand, of form

$$u(t) = y(t) = \alpha p(t) + \beta.$$

and that h is such that $|h/(\alpha - a)| < 1$. Obtain a formula for $p(t)$.

By subtraction we have

$$(\alpha - a)p(t) + (\beta - b) - hp(t - r) = 0$$

or

$$p(t) = \mu + \frac{h}{\alpha - a}p(t - r),$$

with

$$\mu = -\frac{\beta - b}{\alpha - a}$$

But then

$$p(t - r) = \mu + \frac{h}{\alpha - a}p(t - 2r)$$

and by substitution

$$p(t) = \mu + \frac{h}{\alpha - a}\mu + \left(\frac{h}{\alpha - a}\right)^2 p(t - 2r)$$

But also

$$p(t - 2r) = \mu + \frac{h}{\alpha - a}p(t - 3r)$$

and this also may be substituted. By continuing this process a solution is obtained. It is the only solution which has remained bounded, say in absolute value $<p$, for all previous values of t.

3. Consider the same situation except that b is replaced by a given function $b(t)$ of t, such that $b(t) <$ some constant B. Thus if $b(t) = b \sin kt$ we have an example of a periodic demand law, corresponding to possible effects of the seasons.

CHAPTER V

CHANGES IN THE COST AND DEMAND CURVES. TAXATION

28. A Tax on Monopoly.—Although we have heretofore limited ourselves to functions which are simply related to a single variable p, the results of our analysis may be somewhat extended by considering more in detail the relation of the various quantities to the coefficients in the cost and demand curves. Thus we can investigate phenomena which may be ascribed to changes in these coefficients. In particular, many forms of taxation may be so regarded, although we must distinguish between general taxes which affect the distribution of income in the whole community, and special taxes, levied against single commodities, which may to a first approximation at any rate, be assumed to leave substantially unchanged the conditions outside of this commodity. We may suppose that the price of this commodity alone is sensibly affected.

In order to get a concrete result, let us suppose that the algebraic signs of A, B, C, a, b, are as before, and that a special tax of value ξ per unit of production is levied against a certain monopolist. That is the same as increasing the cost of production per unit time by ξu, so that the cost function becomes

$$q(u) = Au^2 + (B + \xi)u + C,$$

as if B were increased by ξ. Hence if the profit is to be a maximum under the new conditions, the new price will be

$$p_1^{(m)} = \frac{b - 2Aab - (B + \xi)a}{-2a(1 - Aa)}$$

and the new amount produced in unit time

$$u_1^{(m)} = \frac{b + (B + \xi)a}{2 - 2Aa}$$

so that the respective changes are

$$\Delta p^{(m)} = \frac{\xi}{(2 - 2Aa)} \text{ and } \Delta u^{(m)} = \frac{\xi a}{(2 - 2Aa)} = a\Delta p. \qquad (1)$$

Hence, ξ being positive, p is increased and u decreased.

Consider now the change in the profit for the monopolist. The new profit is $(p + \Delta p)(u + \Delta u) - A(u + \Delta u)^2 - (B + \xi)(u + \Delta u) - C$, and therefore the change is

$$\Delta\pi = p\Delta u + u\Delta p + \Delta p\Delta u - 2Au\Delta u - A\Delta u^2 - B\Delta u - \xi u - \xi\Delta u, \quad (2)$$

in which p and u represent the old price and amount produced per unit time. We may however at this point introduce a simplification in this expression by assuming that ξ is so small that terms involving ξ^2 are negligible in comparison with terms involving ξ. Hence, since Δu and Δp involve ξ to the first power, we get the following approximate formula for the change in profit:

$$\delta\pi = p\Delta u + u\Delta p - 2Au\Delta u - B\Delta u - \xi u$$

and this reduces to

$$\delta\pi = -\xi\frac{b + Ba}{2 - 2Aa} = -\xi u, \quad (3)$$

on substituting the values of u, p, Δu, Δp. If we take into account the actual instead of the approximate value of the profit, equation (2) yields

$$\Delta\pi = -\xi u_1 + (1 - Aa)\frac{\Delta u^2}{a} = -\xi u_1 + \frac{\xi}{2}\Delta u \quad (3.1)$$

On the other hand the revenue from the tax is the quantity

$$r = \xi u_1,$$

whence

$$\Delta\pi + r = \frac{\xi\Delta u}{2},$$

which is essentially negative, although a small quantity of the second order. In other words, the revenue from the tax does not quite make up the loss of profit to the producer, in spite of the fact, as is shown in equation (1), that the change of price caused by the tax is less than half of the tax itself.

The result just given concerns the producer. On the other hand, the consumer also may be regarded as suffering a loss. For he pays more for the amount u_1 than he would have paid for the same amount before; and if he wishes to resell, the selling value of what he takes after the tax is less than that of the amount

u at the price p. The increase in money value of the amount u_1, that is, the increased amount the consumers would pay for the amount u_1, bought in unit time, over what would have been paid for the same amount if the price were p, is $u_1\Delta p$, or the quantity

$$\frac{r}{2 - 2Aa},$$

which is less than half of the revenue from the tax. On the other hand the consumers pay less for the new amount at the new price than they paid for the old amount at the old price; in fact the gain in selling value is approximately

$$p\Delta u + u\Delta p = (ap + u)\Delta p = (2u - b)\Delta p$$

or

$$\delta(pu) = a\Delta p\frac{B + Ab}{1 - Aa} \tag{4}$$

a quantity which is essentially negative, since $B > 0$, $A > 0$.

In this way we get an idea of the effect of the tax in its relative incidence on producer and consumer. Nevertheless we must guard against assuming that the discussion is complete, for we have made no hypothesis about the use of the revenue or of the consumer's use of the amount $pu - p_1u_1$.

EXERCISE 1.—Show graphically that $\delta(pu)$ is negative.

EXERCISE 2.—Obtain the simple expressions which result in the case when the cost curve is a straight line, that is, when $A = 0$. Interpret the results of this section when $A < 0$, but $Aa < 1$.

29. Small Changes of Functions.—The analysis of the previous section, by means of which a tax was taken account of by a change in the coefficient B, suggests the advisability of considering such changes in general, supposing them small. Here the method of procedure is as old as the calculus. We make use of the fact, on which the definition of derivative is founded, that if $f(x)$ is a function with a derivative, its change for a small change in x is approximately the value of the derivative multiplied by the change in x. This fact is conveniently written in the form

$$df = \frac{df}{dx}dx$$

where df is approximately the change in $f(x)$ and dx is the change in x. (See Fig. 20.)

If we have a function $\varphi(x, y, z,)$ which involves other variables besides x we can, on the same hypothesis, write the change of φ, due to changing x alone by a small amount, as $(\partial\varphi/\partial x)dx$.

$$d_x\varphi = \frac{\partial\varphi}{\partial x}dx$$

But in order to go further we must assume more about the function than merely that it has derivatives with respect to the various variables. We assume that if the variables are changing together by various small amounts dx, dy, dz, the change of the function φ will be obtained approximately by adding together the changes of φ when each variable separately makes its change. This idea may be expressed in symbols as follows:

$$d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = d_x\varphi + d_y\varphi + d_z\varphi. \tag{5}$$

The meaning of saying that $d\varphi$ is approximately the change of φ, which is ordinarily denoted by $\Delta\varphi$, is that the difference $d\varphi - \Delta\varphi$ is small in comparison with the largest of the small quantities dx, dy, and dz; in fact if we denote by Δ the largest of the quantities dx, dy, dz the ratio $|d\varphi - \Delta\varphi|/\Delta$ itself approaches zero as Δ approaches zero. In other words the smaller Δ is, the more insignificant in proportion is the difference $d\varphi - \Delta\varphi$.

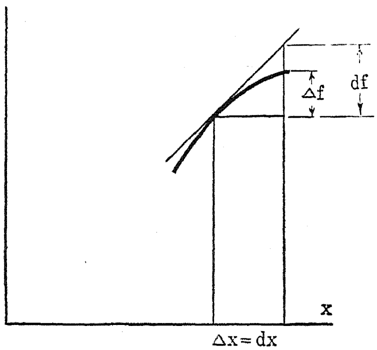

FIG. 20.

The property just described follows from other simple properties of functions. It is unnecessary to discuss these properties here since the equation (5) can easily be verified directly for such functions as polynomials and their ratios, with which we are mainly concerned. Moreover that method of calculating small changes is familiar in economic statistics, and therefore it is not too much to assume that the same method will apply to the functions which represent the statistics in approximate fashion.

EXERCISE 1.—The volume of a sphere is $4\pi r^3/3$. What is the change in volume when r changes from 10 to 10.01?

$dV = (\partial V/\partial r)dr = 4\pi r^2 dr = 4\pi(10)^2(.01) = 12$, approximately.

EXERCISE 2.—The volume of a cylinder is $\pi r^2 h$. What is the change in volume when r changes from 10 to 10.01 and h changes from 20 to 21?

$$dV = \left(\frac{\partial V}{\partial r}\right)dr + \left(\frac{\partial V}{\partial h}\right)dh, \text{ approximately,}$$
$$= 2\pi rhdr + \pi r^2 dh$$
$$= 2\pi(10)(20)(.01) + \pi(10)^2(1) = 4\pi + 100\pi = 104\pi.$$

EXERCISE 3.—The cost of making a thousand tomato cans is given by a formula of the following sort:

$$q = k(2r^2 + rh),$$

where r is the radius of the base and h is the height. What is the change in cost if the height is increased from 4 inches to 4.5 inches and the diameter of the base from 4 to 4.5 inches?

EXERCISE 4.—What is the change in the length of the equator if it is put on a trestle 100 ft. high all around the earth?

EXERCISE 5.—What is the change in the length of the hypotenuse of a right triangle whose sides change by the same small fraction of their original lengths?

30. Changes in the Coefficients of the Cost and Demand Functions.—In this section we deal with the changes in p, u, π, etc., when the coefficients A, B, C, a, b, are slightly changed, in accordance with equation (5). As is evident from (5) the calculation depends upon determining the derivatives of the quantity under investigation with regard to the A, B, C, a, b, as variables. Most of these quantities may be written simply in terms of the total production per unit time under the conditions of equilibrium, and therefore it is desirable to make a table of the derivatives of $u^{(m)}$, $u^{(c)}$, etc., as given by the various postulates.

For

$$u^{(m)} = \frac{b + Ba}{2(1 - Aa)}$$

we have, for example

$$\frac{\partial u^{(m)}}{\partial A} = -\frac{b + Ba}{2(1 - Aa)^2}(-a) = a \cdot \frac{b + Ba}{2(1 - Aa)^2} = \frac{au^{(m)}}{1 - Aa}$$

In a similar fashion

$$\frac{\partial u^{(m)}}{\partial a} = \frac{(b + Ba)A}{2(1 - Aa)^2} + \frac{B}{2(1 - Aa)}$$

and this may be written in either of the forms

$$\frac{B + Ab}{2(1 - Aa)^2} \text{ or } \frac{B + 2Au^{(m)}}{2(1 - Aa)}$$

TABLE I

	$\frac{\partial}{\partial A}$	$\frac{\partial}{\partial B}$	$\frac{\partial}{\partial C}$	$\frac{\partial}{\partial a}$	$\frac{\partial}{\partial b}$	$\frac{\partial}{\partial n}$
$u^{(m)} = \frac{b + Ba}{2(1 - Aa)}$	$a \frac{b + Ba}{2(1 - Aa)^2} = a \frac{u^{(m)}}{1 - Aa}$	$\frac{a}{2(1 - Aa)}$	0	$\frac{B + Ab}{2(1 - Aa)^2} = \frac{B + 2Au^{(m)}}{2(1 - Aa)}$	$\frac{1}{2(1 - Aa)}$	
$u_i^{(c)} = \frac{b + Ba}{2n - 2Aa}$	$a \frac{b + Ba}{2(n - Aa)^2} = a \frac{u_i^{(c)}}{n - Aa}$	$\frac{a}{2(n - Aa)}$	0	$\frac{nB + Ab}{2(n - Aa)^2} = \frac{B + 2Au_i^{(c)}}{2(n - Aa)}$	$\frac{1}{2(n - Aa)}$	$-\frac{u_i^{(c)}}{n - Aa}$
$u_i^{(a)} = \frac{b + Ba}{n + 1 - 2Aa}$	$2a \frac{b + Ba}{(n + 1 - 2Aa)^2} = \frac{2au_i^{(a)}}{n + 1 - 2Aa}$	$\frac{a}{n + 1 - 2Aa}$	0	$\frac{(n + 1)B + 2Ab}{(n + 1 - 2Aa)^2} = \frac{B + 2Au_i^{(a)}}{n + 1 - 2Aa}$	$\frac{1}{(n + 1 - 2Aa)}$	$-\frac{u_i^{(a)}}{n + 1 - 2Aa}$
$u_i^{(b)} = \frac{b + Ba}{n - 2Aa}$	$2a \frac{b + Ba}{(n - 2Aa)^2} = \frac{2au_i^{(b)}}{n - 2Aa}$	$\frac{a}{n - 2Aa}$	0	$\frac{nB + 2Ab}{(n - 2Aa)^2} = \frac{B + 2Au_i^{(b)}}{n - 2Aa}$	$\frac{1}{(n - 2Aa)}$	$-\frac{u_i^{(b)}}{n - 2Aa}$

For n constant and all forms of production:

$$dp = \frac{andu_i - adb - (nu_i - b)da}{a^2}, \quad du = ndu_i,$$

$$d\pi_i = \frac{[2(n - Aa)u_i - (b + Ba)]du_i - au_i^2dA - au_idB - u_idb}{a} + \frac{bu_i - nu_i^2}{a^2}da - dC$$

$$d\pi = nd\pi_i.$$

Corresponding formulae are obtained for the other derivatives, as indicated in Table I.

Moreover since $p = \frac{u - b}{a} = \frac{nu_i - b}{a}$ in all cases, we have the general formula (n being constant)

$$dp = \frac{ndu_i - db}{a} - \frac{(nu_i - b)da}{a^2} = \frac{andu_i - adb - (nu_i - b)da}{a^2}$$

and since

$$\begin{aligned}\pi_i &= pu_i - q(u_i) = pu_i - Au_i^2 - Bu_i - C \\ &= \frac{nu_i^2 - bu_i}{a} - Au_i^2 - Bu_i - C = \\ &\qquad \frac{(n - Aa)u_i^2 - (b + Ba)u_i}{a} - C\end{aligned}$$

we have in all cases (n constant),

$$d\pi_i = \frac{[2(n - Aa)u_i - (b + Ba)]du_i - au_i^2dA - au_idB}{a} + \frac{bu_i - nu_i^2}{a^2}da - \frac{u_idb}{a} - dC.$$

EXERCISE.—Obtain the corresponding formulae when n is large.

31. Unobstructive Taxes.—The tax studied at the beginning of this chapter was equivalent to a change in the constant B of the cost curve, and thus led to a change both in the price and the amount produced, under monopoly conditions. On the other hand, since the constant C does not enter into either the formula for u or p, a tax which is equivalent to a change in C will not affect these equilibrium values. Thus a tax on the capital value of a factory diminishes the profit, but being merely a change in the overhead cost, has no effect on the optimum values for production and price.

But here a word of caution is necessary. It is only as applied to a special tax that this argument is valid. Taxes on capital value, applied equally to all kinds of property, affect the income of a large portion of the community, and thereby influence the demand law of the particular commodity, and lead to changes in the a, b, as well as the C. An assumption about the nature of such changes would perhaps carry us outside our present range of hypotheses, since it would have to be based on relations which involve more than a single commodity. It follows

therefore that for a tax to come under the principle of this section, it must be one which is created for a particular industry, and one of not too great magnitude. It is probably the reverse phenomenon which is the more frequent—namely, a special exemption from a capital tax—which is equivalent to a decrease in the value of C, but of course also has no influence on price or rate of production.

A second form of unobstructive tax, as we may call it, is one levied on a particular kind of income. Suppose, in fact, that a tax of ξ dollars is levied on every dollar of profit of the production of a particular commodity. If π_i is retained as a symbol for the expression $pu_i - q(u_i)$, the profit becomes

$$\pi_i' = (1 - \xi)\pi_i.$$

But since one function is a positive constant multiplied by the other, the same values of u_i and p that make π_i a maximum make also π_i' a maximum, and therefore the optimum values of u_i and p are not affected.

This argument, for the reason previously given, does not apply to a general income tax. If we look for a place where the hypotheses may be realized in practice, we find something similar perhaps, in the surtax. This is an additional tax applied to certain classes of incomes, say those that exceed a stated amount π_0. The resultant income is then given by

$$\pi_i' = \pi_i - \xi(\pi_i - \pi_0), \tag{6}$$

if, for simplicity, we neglect other changes in π, according to the principle of equation (5). But from (6),

$$\pi_i' = (1 - \xi)\pi_i + \xi\pi_0$$

which is a maximum when π_i is a maximum.

32. Tax on Money Value of Production.—Consider again a special tax, that is, one levied on a particular commodity, but assume this time that the tax is of amount ξ on each unit of value produced. The tax will amount to ξp on each unit of commodity produced if the price is p, and therefore the total tax will be ξpu per unit time. Let us apply the tax to a monopoly, as in Sec. **28.** The new function for the profit will be

$$\bar{\pi} = pu(1 - \xi) - Au^2 - Bu - C$$

or, by means of the relation $u = ap + b$,

$$\bar{\pi} = \frac{u^2 - ub}{a}(1 - \xi) - Au^2 - Bu - C$$

But this may be written as

$$\bar{\pi} = \frac{u^2 - ub}{\bar{a}} - Au^2 - Bu - C \tag{7}$$

if $\bar{a} = a/(1 - \xi)$, or, to a first approximation,[1]

$$\bar{a} = a(1 + \xi) = a + \xi a. \tag{8}$$

Hence the value of u which will make $\bar{\pi}$ a maximum may be obtained by substituting $\bar{a} = a + a\xi$ for a in $u^{(m)}$. We obtain then (see Table I, Sec. **30**)

$$du^{(m)} = \frac{B + Ab}{2(1 - Aa)^2} \cdot da = \frac{B + Ab}{2(1 - Aa)^2} \cdot a\xi. \tag{9}$$

The price is related to the new production by the demand equation $u = ap + b$ with a unchanged; hence $du^{(m)} = adp^{(m)}$, and

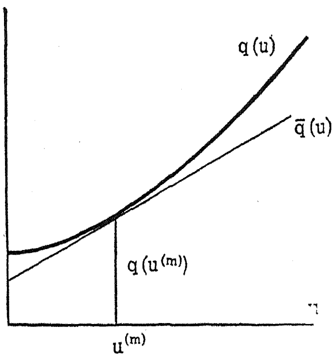

FIG. 21.

$$dp^{(m)} = \frac{B + Ab}{2\ (1 - Aa)^2} \cdot \xi. \tag{9.1}$$

If we use the second form for $\partial u^{(m)}/\partial a$, we obtain the equivalent formula

$$dp^{(m)} = \frac{B + 2Au^{(m)}}{2(1 - Aa)} \cdot \xi = \frac{q'(u^{(m)})}{2(1 - Aa)} \cdot \xi \tag{9.2}$$

These formulae yield the value

$$dp^{(m)} = \tfrac{1}{2}B\xi, \tag{10}$$

in the special situation where A happens to be zero.

Equation (10) justifies a slight digression. Suppose that we replace the cost curve by the straight line which sticks most closely to it in the neighborhood of the value $u^{(m)}$. This replaces $q(u)$ by the linear function (Fig. 21)

$$\bar{q}(u) = q'(u^{(m)})(u - u^{(m)}) + q(u^{(m)}),$$

which, upon evaluation, is seen to be given by the formula

$$\bar{q}(u) = \frac{Ab + B}{1 - Aa}u + \left[C - A\frac{(b + Ba)^2}{4(1 - Aa)^2}\right], \tag{11}$$

[1] The relation

$$\frac{1}{1 - \xi} = 1 + \xi + \xi^2 + \cdots$$

is valid if $|\xi| < |1$.

in which the coefficient of u is

$$\overline{B} = \frac{Ab + B}{1 - Aa}.$$

Since the new cost curve is parallel to the selling value curve for $u = u^{(m)}$, the change from the quadratic function to this special linear function does not change the position of equilibrium, or the value of $p^{(m)}$.

What would be now the effect of a tax on $p^{(m)}$ if we had the linear cost function (11)? Equation (10) yields the value

$$dp^{(m)} = \frac{B\xi}{2} = \frac{Ab + B}{2(1 - Aa)} \cdot \xi \tag{11.1}$$

This value would be greater than the value given by (9.1), since $1 - Aa > 1$. In fact it is the curvature of the cost curve which accounts for the extra factor $(1 - Aa)$ in the denominator of (9.1).

Let us now return to the more general situation and determine the change in π. We have

$$d\pi = d(p^{(m)}u^{(m)}) - d(Au^{(m)2}) - d(Bu^{(m)}) - dC - \xi p^{(m)}u^{(m)}$$

to small quantities of the first order. This gives us

$$d\pi = (p^{(m)} - 2Au^{(m)} - B)du^{(m)} + u^{(m)}dp^{(m)} - \xi p^{(m)}u^{(m)},$$

which reduces, on substitution of the values of $u^{(m)}$, $p^{(m)}$ merely to the last term

$$d\pi = -\xi p^{(m)}u^{(m)}, \tag{12}$$

as a first approximation. In other words, to a first approximation, the loss to the producer is the amount of revenue produced by the tax.

It will be noticed that *for a monopoly the loss in profit due to the imposition of a special tax of any of the three kinds considered is to a first order of approximation equal to the revenue produced by the tax.* This theorem holds independently of the algebraic signs of the coefficients of our cost function. If further A is positive, as we have assumed it heretofore, the loss to the producer exceeds the revenue by a small amount, for the kinds of taxation considered in Sec. **28** and **32.**

Another remark in regard to the coefficients may well be made. The equilibrium prices are fixed by a relation between the slopes of the cost curves and the selling value curves. For a small change of u_i, the change of the slope of the cost curve enters as a

small quantity of the same order—as we saw above by considering a linear cost function— but not any further derivatives even if the cost function is absolutely general. In other words, the results established in this chapter apply to any sort of a cost function, $2A$ taking the place of $q''(u)$ in the neighborhood of the optimum values under consideration. But this remark does not apply to the demand function in the same fashion.

33. Continuation. Cooperation and Competition.—If we apply the tax of Sec. **32** to an industry where the determining postulate is that of cooperation, the new value of $u_i^{(c)}$ may be obtained from the old by the substitution of $\bar{a} = a + a\xi$, as in Sec. **32.** This yields the result

$$du_i^{(c)} = \frac{nB + Ab}{2(n - Aa)^2} \cdot a\xi, \text{ whence } dp^{(c)} = \frac{nB + Ab}{2(n - Aa)^2} \cdot n\xi, \tag{13}$$

and by substitution in the formula

$$\begin{aligned} d\pi_i &= pdu_i + u_i dp - 2Au_i du_i - Bdu_i - \xi u_i p \\ &= (p - 2Au_i - B)du_i + u_i dp - \xi u_i p_i, \end{aligned}$$

remembering that $du_i = adp/n$, since $nu_i = ap + b$, we obtain finally $d\pi_i = -\xi u_i^{(c)} p^{(c)}$, so that we have

$$d\pi = -\xi u^{(c)} p^{(c)}. \tag{13.1}$$

This result is analogous to those obtained for monopoly, in that to a first approximation we have $d\pi + r = 0$.

In the case of competition of the kind (a) the situation is different. In that case we have

$$du_i^{(a)} = \frac{(n + 1)B + 2Ab}{(n + 1 - 2Aa)^2} \cdot a\xi, \quad dp^{(a)} = \frac{(n + 1)B + 2Ab}{(n + 1 - 2Aa)^2} \cdot n\xi, \tag{14}$$

and the formula for $d\pi_i$ reduces to the value

$$d\pi_i = \frac{n - 1}{n} u_i^{(a)} dp^{(a)} - \xi u_i^{(a)} p^{(a)}, \text{ with } d\pi = nd\pi_i$$

Hence

$$r + d\pi = \frac{n - 1}{n} u^{(a)} dp^{(a)}.$$

But this result is essentially positive, and of the same order of small quantities as ξ. The loss to the producer is in this case less than the revenue from the tax, even as a first approximation.

34. General Exercises.

1. For what value of ξ in the problem of Sec. **28** will the revenue r be a maximum?

Suggestion.—First obtain a formula for r in which ξ is the only variable.

2. If the coefficients A, B, C change, the cost function itself changes to a new function $(A + dA)u^2 + (B + dB)u + C + dC$. If we denote the change in this function for an arbitrary value of u by $\delta q(u)$, the function $\delta q(u)$ is given by the formula

$$\delta q(u) = dAu^2 + dBu + dC$$

Similarly, for the change in the marginal unit cost function, we have the formula

$$\delta q'(u) = 2dAu + dB$$

Show that if A, B, C change, but not a, b, the change in $p^{(m)}$ is given by the formula

$$dp^{(m)} = \frac{\delta q'(u^{(m)})}{2 - 2Aa}.$$

3. Following the method of problem 2, show that

$$dp^{(c)} = n\frac{\delta q'(u_i^{(c)})}{2(n - Aa)},$$
$$dp^{(a)} = n\frac{\delta q'(u_i^{(a)})}{n + 1 - 2Aa}, \quad dp^{(b)} = n\frac{\delta q'(u_i^{(b)})}{n - 2Aa}.$$

4. With the aid of the results of the previous problem, discuss the effect in the several cases of a tax of amount ξ per unit of quantity produced.

5. Find the dimensions of ξ as used in Secs. **28** and **32**.

6. Find the $du^{(m)}$ of Sec. **32** by using proper values of dA and dB, deduced from the formula for π.

7. Discuss the effect of a tax like that of Sec. **32** on an industry subject to competition of type (b). Show incidentally that

$$du_i^{(b)} = \frac{nB + 2Ab}{(n - 2Aa)^2}a\xi, \quad dp = \frac{nB + 2Ab}{(n - 2Aa)^2}n\xi$$
$$d\pi_i = u_i^{(b)}dp - \xi p^{(b)}u_i^{(b)},$$

and that $r + nd\pi_i$ is not zero approximately, but essentially positive. Consider particularly the case where $A = 0$.

8. Show by means of the graphs that in the monopoly problem a change in the cost curve to one which is everywhere steeper produces a decrease in the optimum value of u. Thus justify qualitatively the result in problem 2. Assume that the slope of the cost curve is continually increasing and that of the selling value curve continually decreasing in the neighborhood of the values in question.

9. Do the results obtained in this chapter apply to any sort of demand function, interpreting a as du/dp?

10. Discuss the meaning of the results of exercises 3, 4, 6, 7 if A is no longer required to be positive, but is allowed to be negative.

CHAPTER VI

DIVERSIFICATION OF COST. TARIFF. RENT

35. Introduction.—The tariff is an assessment against industry which discriminates against one competitor in favor of another, increasing the obligations against the first. In that way it is essentially different from taxation, which, from a political point of view is supposed to be levied more or less equally against all classes. The new question therefore, instead of being incidental to the consideration of the sort of changes in the cost function so far discussed, becomes merged in the problem of determining the effect of changes in the cost functions which are different among different producers.

If we have n producers, and wish to change all their cost functions by small amounts, we can make use of our fundamental assumptions for functions of several variables, and obtain the result by changing each cost function separately. Suppose then that we are dealing with n producers, the last $n - 1$ of them having the same cost function $Au_i^2 + Bu_i + C$ but the first producer being subject to one with slightly different coefficients, $A + dA$, $B + dB$, $C + dC$.

36. Cooperation and Competition.—Consider first the case of cooperation. The formula for the profit becomes

$$\pi = p\sum_1^n u_k - A\sum_1^n u_k^2 - B\sum_1^n u_k - nC - u_1^2 dA - u_1 dB - dC \quad (1)$$

and the equations which determine the situation of equilibrium are the following:

$$\begin{aligned} \frac{\partial \pi}{\partial u_i} &= 0 = p + \frac{1}{a}\sum_1^n u_k - 2Au_i - B, \qquad i = 2, 3, \cdots n, \\ \frac{\partial \pi}{\partial u_1} &= 0 = p + \frac{1}{a}\sum_1^n u_k - 2Au_1 - B - 2u_1 dA - dB, \end{aligned} \quad (2)$$

since we have $\partial p/\partial u_i = 1/a$ from the equation $\sum_1^n u_i = ap + b$.

But if dA and dB are small the changes in the u_i and the p will be small, so that $p = p^{(c)} + dp$, $u_i = u_i^{(c)} + du_i$, where the $u_i^{(c)}$, $p^{(c)}$ are given by their old formulae in terms of A, B, n. Now these old values satisfy the equations (2) when the dA and dB vanish, and therefore if we substitute $p = p^{(c)} + dp$, etc., a great many terms will cancel against each other. In fact, if we drop products like $dA\,du_i$, which are small in comparison to the others, we shall have left the following simple set of equations:

$$dp + \frac{1}{a}\sum_{1}^{n} du_k - 2A\,du_i = 0, i = 2, 3, \cdots, n,$$

$$dp + \frac{1}{a}\sum_{1}^{n} du_k - 2A\,du_1 = 2u_1dA + dB$$

$$\sum_{1}^{n} du_k = a\,dp$$

which again are merely the following:

$$\begin{aligned} 2dp - 2A\,du_i &= 0, \quad i = 2, 3, \cdots, n, \\ 2dp - 2A\,du_1 &= 2u_1dA + dB. \end{aligned} \tag{3}$$

And finally, if we sum all of the equations (3) we obtain the relation

$$2n\,dp - 2Aa\,dp = 2u_1dA + dB.$$

Whence

$$dp = \frac{2u_1dA + dB}{2n - 2Aa},$$

where $u_1 = \dfrac{b + Ba}{2n - 2Aa}$. From (3), then

$$du_i = \frac{1}{A}\,dp, \quad i = 2, 3, \cdots, n, \tag{4.1}$$

and from $\sum_{1}^{n} du_i = a\,dp$,

$$du_1 = -\frac{(n - 1 - Aa)dp}{A} \tag{4.2}$$

In the case of competition of kind (a) we have

$$\begin{aligned} \pi_i &= pu_i - Au_i^2 - Bu_i - C, \quad i = 2, \cdots, n, \\ \pi_1 &= pu_1 - (A + dA)u_1^2 - (B + dB)u_1 - (C + dC), \end{aligned}$$

and the state of equilibrium is determined by the equations

$$\frac{\partial \pi_i}{\partial u_i} = 0 = p + \frac{u_i}{a} - 2Au_i - B, \quad i = 2, 3, \cdots, n,$$

$$\frac{\partial \pi_1}{\partial u_1} = 0 = p + \frac{u_1}{a} - 2Au_1 - B - 2u_1 dA - dB.$$

If again we consider merely the differentials, dp, du_i writing $p = p^{(a)} + dp$, etc., all terms will cancel from the above equations which do not contain differentials, dp, du, or dA, dB, since the original values satisfy those equations in their original form. We have therefore, retaining merely terms of the first order of small quantities,

$$\begin{aligned} dp + \frac{du_i}{a} - 2A du_i &= 0, \quad i = 2, 3, \cdots n, \\ dp + \frac{du_1}{a} - 2A du_1 &= 2u_1 dA + dB \\ \sum_1^n du_k &= a dp \end{aligned} \tag{5}$$

whence

$$ndp + \frac{1}{a}\sum_1^n du_k - 2A\sum_1^n du_k = 2u_1 dA + dB$$

or

$$(n + 1 - 2Aa)dp = 2u_1 dA + dB,$$

from which we obtain the value of dp:

$$dp = \frac{2u_1 dA + dB}{n + 1 - 2Aa}, \text{ with } u_1 = \frac{b + Ba}{n + 1 - 2Aa}. \tag{6}$$

From the first of equations (5), we have

$$du_i = \frac{-a}{1 - 2Aa} dp, \tag{6.1}$$

and finally

$$du_1 = \frac{n - 2Aa}{1 - 2Aa} a dp. \tag{6.2}$$

A similar calculation for competition of kind (*b*) yields the results

$$\begin{aligned} dp &= \frac{2u_1 dA + dB}{n - 2Aa}, \text{ with } u_1 = \frac{b + Ba}{n - 2Aa}, \\ du_i &= \frac{dp}{2A}, \quad i = 2, \cdots n \\ du_1 &= -\frac{n - 1 - 2Aa}{2A} dp. \end{aligned} \tag{7}$$

It will be noticed that in all of the cases just treated the du_i, $i = 2, 3, \cdots, n$, have the same algebraic sign as dp, while the du_1, which corresponds to the producer whose cost function has been changed, is of opposite algebraic sign to dp. This result of course depends on the assumption that $A > 0$. Moreover it is seen that there is a striking relation between the changes in cooperation and in competition (*b*), given by comparison of the equations (4.1) and the second of equations (7).

37. Simultaneous Changes.—If we make simultaneous changes dA_1, dB_1, dC_1, in the coefficients of the first cost function and dA_2, dB_2, dC_2, in the coefficients of the second cost functions, we can obtain the resulting values of dp, etc., in accordance with our principle for the calculation of the change of a function of several variables. In fact, we have immediately, taking the case of competition (*b*) for illustration,

$$dp = dp_1 + dp_2, \tag{8}$$

where

$$dp_1 = \frac{2u_1dA_1 + dB_1}{n - 2Aa}, \quad dp_2 = \frac{2u_1dA_2 + dB_2}{n - 2Aa}, \quad u_1 = \frac{b + Ba}{n - 2Aa}$$

and

$$\begin{aligned} du_1 &= -\frac{n - 1 - 2Aa}{A}dp_1 + \frac{dp_2}{2A} \\ du_2 &= \frac{dp_1}{2A} - \frac{n - 1 - 2Aa}{A}dp_2 \\ du_i &= \frac{dp_1 + dp_2}{2A}, \quad i = 3, 4, \cdots n. \end{aligned} \tag{8.1}$$

In particular if all the cost function coefficients are changed by the same amounts dA, dB, this method yields the results

$$dp = n\frac{2u_1dA + dB}{n - 2Aa}$$

$$du_i = \frac{(n - 1)dp}{2An} - \frac{(n - 1 - 2Aa)dp}{2An} = \frac{adp}{n},$$

and these agree precisely, as they should, with the formula of example (3) of the previous chapter.

38. Large Changes.—The formulae so far obtained admit only relatively small values of dA, dB. A tariff is often large. Moreover the usual reason for imposing a tariff is a wide difference in two classes of cost functions. Hence it is desirable to obtain formulae like those given in Chapter III, for widely divergent values of the A, B, C. For the case of cooperation and com-

petition (*b*) that method is entirely satisfactory and may be extended to an arbitrary number of producers. For the other kind of competition, the formulae are not so simple, and it is perhaps desirable to consider the changes as generated by the successive application of small changes, that is, by a process of integration.

Suppose we have n cost functions $A_k u_k^2 + B_k u_k + C_k$ and change the coefficients by small amounts dA_k, dB_k, dC_k. Then

$$\pi_i = pu_i - A_i u_i^2 - B_i u_i - C_i - dA_i u_i^2 - dB_i u_i - dC_i$$

and

$$\frac{\partial \pi_i}{\partial u_i} = 0 = p + \frac{u_i}{a} - 2A_i u_i - B_i - 2u_i dA_i - dB_i, \qquad i = 1, 2, \cdots, n. \quad (9)$$

if we denote the values of u_i, p before changing the coefficients by u_i, p, and after the change by $u_i + du_i$, $p + dp$, the equations in the differentials become

$$dp + \frac{(1 - 2A_i a)du_i}{a} = 2dA_i u_i + dB_i$$

or

$$\frac{dp}{1 - 2A_i a} + \frac{du_i}{a} = \frac{2dA_i u_i + dB_i}{1 - 2A_i a}, \quad i = 1, 2, \cdots, n. \quad (10)$$

If we add all these equations together, remembering that $\sum_1^n du_i = adp$, we have

$$dp\left(\sum_1^n \frac{1}{1 - 2A_i a} + 1\right) = \sum_1^n \frac{2dA_i u_i + dB_i}{1 - 2A_i a},$$

or

$$dp = \frac{\displaystyle\sum_1^n \frac{2dA_i u_i + dB_i}{1 - 2A_i a},}{1 + \displaystyle\sum_1^n \frac{1}{1 - 2A_i a}} \quad (11)$$

and this with equations (9) form a set of differential equations to determine p, and $u_1, \ldots, u_n$.

The equations (11) need not be solved, since their solutions are already given implicitly by (9). They may however tell us something directly which will not be easily obtainable from (9). In fact, on account of the assumptions that we have made about

the signs of the coefficients, we see directly that if the dA_i, dB_i are all positive, the dp is also; in other words, if the A_i, B_i all increase from arbitrarily given values the p increases also from its corresponding value.

39. Tariffs.—Let us consider a situation where, at the beginning, n producers have identical cost functions, and let us impose a small tariff of ξ dollars on each dollar of selling value of the production of one of them, say the first. We assume that we are dealing with competition of type (*b*). The profit functions will then be of the form

$$\begin{aligned} \pi_1 &= pu_1(1-\xi) - Au_1^2 - Bu_1 - C \\ \pi_i &= pu_i - Au_i^2 - Bu_i - C, \quad i = 2, \cdots, n. \end{aligned} \tag{12}$$

Since the quantity $1 - \xi$ is a constant, the function π_1 will satisfy the conditions for a maximum when $\pi_1' = \pi_1/(1-\xi)$ satisfies them, and vice versa. Hence for the purpose of calculating the optimum values of the u_1 and p we can replace π_1 by π_1':

$$\pi_1' = pu_1 - \frac{A}{1-\xi}u_1^2 - \frac{B}{1-\xi}u_1 - \frac{C}{1-\xi}.$$

But as far as small quantities of the first order, we have

$$\frac{A}{1-\xi} = A(1+\xi), \quad \frac{B}{1-\xi} = B(1+\xi)$$

so that our problem can be solved by the artifice of considering A and B in the first profit function replaced by $A + dA$ and $B + dB$ respectively, with $dA = A\xi$, $dB = B\xi$. The formulae (7) then yield immediately the results

$$\begin{aligned} dp &= \frac{2Au_1 + B}{n - 2Aa}\xi = \frac{2Ab + nB}{(n-2Aa)^2}\xi \\ du_i &= \frac{dp}{2A}, \quad i = 2, 3, \cdots, n, \\ du_1 &= -\frac{(n - 1 - 2Aa)dp}{2A}. \end{aligned} \tag{13}$$

Similar methods apply to any of the other forms of organization, which can therefore be treated by the methods of Sec. **36.** Moreover, if the cost functions of the various producers are not initially the same, but have coefficients with arbitrary values, the problem may still be discussed with reference to Sec. **38.** In particular, for cooperation or competition of kind (*b*) the results of the note and exercise of page 25 and the exercise on

page 27 can immediately be generalized to n producers, and the formulae analogous to (13) written down immediately.

40. Rent.—The discussion of rent has perhaps taken an undue place in theories of economics on account of special circumstances. At the time that the classical theories of economics were being first developed, there were two situations which most impressed the English economists: first, that a large part of income from invested capital was in the form of rent on land, and second, that there was in the rapidly expanding America an unlimited quantity of unused land. Hence rent seemed to be a measure, roughly speaking, of the availability of land in cultivation as compared with the more distant or less fertile land that could be obtained for nothing.

For our purposes there is no point in distinguishing between rent on land and periodic payment for the service of other forms of capital. Suppose for instance that there are two producers competing according to a situation of type (*b*), part of the cost function in each case appearing in the constants C_1, C_2 as so much rent paid for the use of a certain form of capital,—money, land, factory, machinery, water rights, etc. Suppose that one of the owners of the form of capital which is rented in the two cases desired to raise the charge asked for the use of that capital in unit time. By how much can he do so?

As far as the specific industry with which we deal is concerned, the question has no bearing, because a change in the C_1 or C_2 has no influence on the price, or amount produced. This will be the case, unless on the one hand the change is great enough to cause not merely a technical loss, but the actual bankruptcy of the producer, or on the other hand, the change in the profit causes the producer to reorganize his production with a different cost function, that is, with different A, B, as well as C, and thus produces a new equilibrium situation.

For the recipient of rent, the situation is that of a producer or owner who sells the services of some definite capital and is therefore like that of any of the producers so far considered. It is the service of the capital that is sold rather than the capital itself. A brief survey of some of the possibilities may be interesting.

In the first place, one may be dealing with the rent of machines protected by patents, where there is all the appearance of a monopoly. Thus, there will be no effective competition in

renting certain patented machinery until the profit of the user is reduced to what it would be without machinery at all.

To say this does not mean that the owner of the capital will necessarily charge enough to cause that result. When we say that rent is a part of the constant C of the cost function of the user of the rented capital, that is indeed the same as saying that the demand for the services of this capital is independent of the price of that service, that is, the rental charge; since the situation of equilibrium of the user is not affected by the amount of this charge. As we have seen then, under monopoly or cooperative conditions for the production of this capital, or even with competition of type (a), the price will be pushed up beyond the point where the hypothesis (that the rent appears as part of C) will remain valid. This is the result established in Sec. **19**, equations (19) and (19.1). But in the case of a good many kinds of machinery installation of new units or removal of superfluous ones is simple enough so that although for any specific situation the cost of using them is a part of the constant C, in the long run the number used is proportional to the number of units of the commodity produced, and will thus in a long term analysis appear as part of B. In other words, in a long term analysis—that is, from the point of view of the owner of the rented machinery—the demand for his product will not be constant but will be effectively of some such form as the expression $ap + b$, with $a \neq 0$; and there will be a definite monopoly value for the rent p.

But this is not the usual case with capital in the form of land or buildings. In fact, the conclusion is now clear, that unless we have something anagolous to comtetition of type (b) in producing the services to be rented, the rental charge will be unlimited by anything except the profits of the user, if it appears in his cost function as part of C—and this without regard to the cost of producing the services to be rented. These latter costs are in fact often negligible, and the income received as rent is "unearned." It is hardly necessary to point out the frequent occurence of this particular economic phenomenon, and how its color has darkened many of the pages of history.

41. General Exercises.

1. Discuss the formulae for cooperation when there are two producers with cost functions $B_1u_1 + C_1$, and $B_2u_2 + C_2$ respectively and $u_1 + u_2 = ap + b$. Show that the equations corresponding to (2) are in general self-contradictory.

In the above case, if $B_1 = B_2$, there is a maximum value of π, but u_1 and u_2 are not determined. If $B_1 \neq B_2$ there is no maximum value if u_1 and u_2 are both positive, not zero; by making one of these quantities zero, however, a maximum value is obtained for π among all values of u_1, u_2 which are ≥ 0. For this value we do not have $\partial\pi/\partial u_1 = 0$, $\partial\pi/\partial u_2 = 0$. Why?

2. Consider the corresponding problem for competition.

CHAPTER VII

ON RATES OF EXCHANGE

42. Transfer of Credit.—In the previous chapters we have been able to treat some simple problems involving the relation of production and demand, without finding it necessary to treat separately the theory of production and the theory of the exchange in the market of the goods produced. But now we wish to focus attention more directly on certain special aspects of the latter theory, and in particular, on the relation of the theory of money to it. It is unnecessary to point out that many, if not most, transactions are carried out with the aid of credit, rather than immediately in currency; and it is natural therefore to consider the mechanism of the transfer of credit. In this chapter we consider the transfer of credit in space, that is, from one market to another, and the consequent determination of rates of exchange. In the next chapter we consider a corresponding transfer in time, and are thus led to a discussion of rates of interest. Finally, we treat the relation of circulating media to the exchange of goods in a single market, and the so-called *equation of exchange*.

We shall do well to follow Irving Fisher in the definition of the fundamental concepts in this subject.[1] Thus we shall define money as "any property right which is generally acceptable in exchange." By means of checks, the bank deposits which are subject to check serve as a medium of exchange in normal times, and we might regard them as generally acceptable in exchange. In fact, the checks on them are the means by which, if we consider total values, the great bulk of exchanges are made. Yet it is convenient to make a distinction between circulating media of this kind, which are acceptable only by consent of the payee, and therefore constitute a rather variable quantity of circulating media, and money proper which, by law or otherwise, the payee accepts without question.

In this chapter we regard circulating media in the general sense of money plus bank deposits subject to check, and consider the

[1] "Purchasing Power of Money," chaps. I, II, New York, 1922.

method by which credit at one place, say New York, may also be regarded as credit at another place, such as London, and what amounts of credit in the two places would be exchangeable against each other. A man who has bank deposits in New York may desire to buy goods in London and pay for them there. Of course he will not want to send gold if he can avoid that expensive process; and therefore he will buy, say, a commercial bill of exchange in New York for the proper amount in English currency, drawn on the London correspondent of the New York bank. This amount will be paid to the payee in London. It may be payable on demand,[1] especially if the bill is transmitted as a cable, or after a definite time, thirty or sixty days or whatever is convenient, if interest is provided for. At suitable intervals of time the various banks may balance their accounts, if necessary, by a shipment of gold.

The question at issue is then not how much money a gold dollar is worth, if one has the gold in London, for this gold would always be exchangeable for the equivalent gold in currency at London, but how much it costs in New York to buy the right to, say, a pound in London. The right to a pound or to x pounds in London is a commodity which is bought and sold in New York, and we wish to get some idea of the price.

EXERCISE.—Sterling exchange being par at \$4.86⅔, that is, there being 4.86⅔ the amount of gold in the pound sterling that there is in the gold dollar, what would it cost to buy the right to 50 pounds sterling in London, if the rate of exchange quoted is 2⅓ cents below par?

43. Reduced Rates of Exchange.—As we have just seen, the problem of determining the price of these rights, which usually differs slightly from the par value, is complicated by various circumstances which tend to obscure the character of the phenomenon with which we are dealing. In the first place there is interest involved except in the case of the demand bills. In the second place the unit of currency is often different in the various markets, and the rate of exchange follows approximately the ratio of units. The question under investigation however is the divergence from that ratio. In the third place, even the value of that ratio is made ambiguous if the bases of currency are not the same, that is, if a mono-metallic country is trading with a bimetallic

[1] The rate of exchange, even on demand bills, is slightly variable, depending on the probable interval of time before they will be presented for payment.

country, or with one which is mono-metallic but based on a different metal, or with one whose money is inflated or otherwise controlled. Let us then in our treatment disregard for the present these circumstances, which are of a subsidiary nature, as far as we are interested in the transfer of credit in space, and assume that all markets are dealing in demand bills, without interest, and are using the same unit and kind of currency. For convenience let us say that the unit is a dollar, redeemable in a fixed weight of gold.

Let now (i), (j) be two markets, and let c_{ij} be the value at (j) of the right to one dollar at (i), that is, the value at (j) of unit credit at (i). In other words, a credit of m dollars at (i) is exchangeable for a credit of mc_{ij} dollars of credit at (j). The number c_{ij} may be called the *reduced rate of exchange* or simply the *rate of exchange* of (i) on (j). From the very definition

$$c_{ji} = \frac{1}{c_{ij}} \text{ or } c_{ij}c_{ji} = 1 \tag{1}$$

In fact, m' dollars credit at (j) is exchangeable with $m'c_{ji}$ dollars of credit at (i), by definition of c_{ji}; but by taking $m' = m/c_{ji}$ we see that m/c_{ji} dollars credit at (j) is exchangeable with m dollars credit at (i). Hence by the previous definition of c_{ij} we have $m/c_{ji} = mc_{ij}$ from which (1) follows. This deduction is an interpretation of the assumption implied by our definition, that the algebraic sign of the transfer does not influence the value of c_{ij}; that is, that transfer of m dollars credit from (i) to (j) goes at the same rate as a transfer of m dollars debit from (i) to (j). The practical nature of this assumption is manifest, since if it were not satisfied, a gain could be obtained by transferring credit in a circle.

The same sort of assumption, if there are more than two markets, leads to the identity

$$c_{ij}c_{jk} = c_{ik} \tag{2}$$

where (i), (j), (k) are any three markets. Equation (2) merely says that credits at (i) and (k) are exchangeable through (j), and that no gain can be made by speculation around the circle (i), (j), (k) in any way. Such speculation, if it goes on, has the effect of keeping these equations valid.

EXERCISE.—The following were the demand rates for foreign exchange on Jan. 8, 1929, for units of foreign currency in terms of American money, and their respective par values:

Great Britain (2)	4.84⅝ dollars,	par 4.8665 dollars
France (3)	3.90¾ cents,	par 3.91¾ cents
Italy (4)	5.23⅜ cents,	par 5.26 cents
Germany (5)	23.75½ cents,	par 23.83 cents.

Denote the American market by (1), and the other markets by the numbers given above, and find the reduced rates c_{21}, c_{12}, c_{31}, c_{13}, c_{23}, c_{45}.

44. Credit Functions.—Let $m_{ij}(t)$ be the sum totals of the values at (i) of credits in the various banks of (i) against the banks of (j); each of these credits is assumed to be positive or zero. The quantity $m_{ij}(t)$ is therefore the total value at (i) of the rights owned by various banks of (i) to deposits and resources of banks at (j). Similarly $m_{ji}(t)$ is the total value at (j) of the rights owned by the various banks of (j) to deposits at (i).

Consider a purchaser of goods at (i), buying them from a seller of some other market, say (j). He pays for them with credit obtainable from (j), it may be, in currency of (i) so that the immediate transaction involves no exchange. On the other hand, the purchaser may pay for the goods by buying credit at (i) against some bank at (j) or at some other market (k), whereupon the seller deposits that credit at (j), or even in some other market (l). The general case is governed by the equation

$$c_{ik}\Delta m_{ik} + c_{lk}\Delta m_{li} = 0 \tag{3}$$

in which Δm_{ik} is negative and Δm_{li} is positive, and the relation states that the credits which are transferred have the same numerical value at the market (k) (and therefore at any other market). The simpler and more usual transaction of those indicated satisfies the equation

$$c_{ij}\Delta m_{ij} + \Delta m_{ji} = 0 \tag{3.1}$$

where the transfer is of credit at (i) to credit at (j).

A more complicated transaction might involve several markets at the same time, but still might be considered as built up of several transactions of simpler type. If we have n markets, and consider all the transactions which involve one market, say the first, we have an equation of the form

$$\Delta m_{12} + \Delta m_{13} + \cdots + \Delta m_{1n} + c_{21}\Delta m_{21} + c_{31}\Delta m_{31} + \cdots + c_{n1}\Delta m_{n1} = 0, \tag{4}$$

which contains equations of type (3), (3.1) as special cases.

If now we divide by Δt and let Δt approach 0, we have the equation

$$\frac{dm_{12}}{dt} + \frac{dm_{13}}{dt} + \cdots + \frac{dm_{1n}}{dt} + c_{21}\frac{dm_{21}}{dt} + c_{31}\frac{dm_{31}}{dt} + \cdots + c_{n1}\frac{dm_{n1}}{dt} = 0.$$

In general then, considering all the markets, we have the n equations

$$0 = \sum_{1}^{n}{}'_{k} \left\{ \frac{dm_{ik}}{dt} + c_{ki}\frac{dm_{ki}}{dt} \right\}, \; i = 1, 2, \cdots n, \tag{5}$$

in which Σ' denotes summation with respect to k where the terms for which $k = i$ are omitted, having no significance. In (5), the c_{ki} as well as the m_{ik}, m_{ki} depend, of course, on the time.

The credit functions $m_{ik}(t)$, just described, have rates of change dm_{ik}/dt which may be either positive or negative since the Δm_{ik} may be either positive or negative. Hence the functions $m_{ik}(t)$ may be sometimes increasing and sometimes decreasing functions of the time. But we can construct, with reference to these, certain functions $M_{ik}(t)$ which are essentially increasing functions of the time and which turn out to have an interesting relation to the theory. They may be described as the "positive variation functions" for the functions $m_{ij}(t)$. Before defining these new credit functions, let us examine more particularly the equations (5).

45. Necessary Relations among Credit Functions.—For simplicity, let us consider the situation where there are only three markets, so that the equations (5) become the following:

$$\begin{aligned}
\frac{dm_{12}}{dt} + \frac{dm_{13}}{dt} + c_{21}\frac{dm_{21}}{dt} + c_{31}\frac{dm_{31}}{dt} &= 0 \\
\frac{dm_{21}}{dt} + \frac{dm_{23}}{dt} + c_{12}\frac{dm_{12}}{dt} + c_{32}\frac{dm_{32}}{dt} &= 0 \\
\frac{dm_{31}}{dt} + \frac{dm_{32}}{dt} + c_{13}\frac{dm_{13}}{dt} + c_{23}\frac{dm_{23}}{dt} &= 0.
\end{aligned} \tag{6}$$

Since the rights to credit in foreign markets may themselves be regarded as commodities, the equations (5) govern also the

equations of speculation in foreign exchange. On the basis of the equations (2)

$$c_{ij}c_{jk} = c_{ik},$$

which we have assumed, we may now eliminate from the equations (6) all but two of the quantities c_{ij}, say c_{21} and c_{31}. To this purpose we multiply the second of equations (6) by c_{21}, and the third by c_{31} and obtain the following equations, after rearranging the terms

$$\begin{aligned} \frac{dm_{21}}{dt}c_{21} + \frac{dm_{31}}{dt}c_{31} &= -\frac{dm_{12}}{dt} - \frac{dm_{13}}{dt} \\ \left(\frac{dm_{21}}{dt} + \frac{dm_{23}}{dt}\right)c_{21} + \frac{dm_{32}}{dt}c_{31} &= -\frac{dm_{12}}{dt} \\ \frac{dm_{23}}{dt}c_{21} + \left(\frac{dm_{31}}{dt} + \frac{dm_{32}}{dt}\right)c_{31} &= -\frac{dm_{13}}{dt}. \end{aligned} \tag{7}$$

But these are three equations which have to be satisfied simultaneously by the two quantities c_{21}, c_{31}.

This last remark implies that the various quantities dm_{21}/dt etc., cannot be independent. In fact the equations (7) are linear in c_{21} and c_{31}. We can therefore easily solve the last two of equations (7), for example, for c_{21} and c_{31} and then substitute the values obtained in the first. The result will then be a relation connecting the various credit functions. When reduced as far as possible, it turns out to have the following form:

$$\frac{dm_{21}}{dt}\frac{dm_{32}}{dt}\frac{dm_{13}}{dt} + \frac{dm_{12}}{dt}\frac{dm_{23}}{dt}\frac{dm_{31}}{dt} = 0 \tag{8}$$

The equation (8) *has to be satisfied, if there are rates of exchange* c_{ij}, $i = 1, 2, 3, j = 1, 2, 3$ *which satisfy the equations* (2). In other words, the process of speculation, which insures the equation (2) may be regarded as keeping credit functions in such adjustment as to satisfy (8).

The easiest way to deduce (8) is to make use of the elementary properties of determinants. In order that the three equations (7) be consistent it is necessary that the determinant of the coefficients (regarding c_{21}, c_{31} as the unknowns) should vanish. If we denote the quantity dm_{ij}/dt by α_{ij} this determinant takes the form

$$\begin{vmatrix} \alpha_{21} & \alpha_{31} & \alpha_{12} + \alpha_{13} \\ \alpha_{21} + \alpha_{23} & \alpha_{32} & \alpha_{12} \\ \alpha_{23} & \alpha_{31} + \alpha_{32} & \alpha_{13} \end{vmatrix} = 0$$

The determinant may be simplified by combining rows, and we have

$$
\begin{aligned}
0 &= \begin{vmatrix} -2\alpha_{23} & -2\alpha_{32} & 0 \\ \alpha_{21} + \alpha_{23} & \alpha_{32} & \alpha_{12} \\ \alpha_{23} & \alpha_{31} + \alpha_{32} & \alpha_{13} \end{vmatrix} \\
&= -2 \begin{vmatrix} \alpha_{21} & \alpha_{32} & 0 \\ \alpha_{23} + \alpha_{23} & \alpha_{32} & \alpha_{12} \\ \alpha_{23} & \alpha_{31} + \alpha_{32} & \alpha_{13} \end{vmatrix} \\
&= -2 \begin{vmatrix} \alpha_{23} & \alpha_{32} & 0 \\ \alpha_{21} & 0 & \alpha_{12} \\ 0 & \alpha_{31} & \alpha_{13} \end{vmatrix} \\
&= \quad 2 \quad (\alpha_{21}\,\alpha_{32}\,\alpha_{13} + \alpha_{23}\,\alpha_{31}\,\alpha_{12}),
\end{aligned}
$$

which yields (8).

EXERCISE.—Express by means of a determinant the identity corresponding to (8) in the case of four markets. What happens in the case of two markets?

46. The Balance of Trade.—We consider again n markets, but for the sake of a simple notation we assume that there is only one bank in each market. The *positive variation function* $M_{ij}(t)$ for the function $m_{ij}(t)$ is defined as a function whose change is identical with the change in $m_{ij}(t)$ while the latter is continually increasing but whose change is zero during the intervals of time when $m_{ij}(t)$ remains constant or decreases. The function $M_{ij}(t)$ will accordingly satisfy the equations

$$
\begin{aligned}
\frac{dM_{ij}(t)}{dt} &= \frac{dm_{ij}(t)}{dt}, \text{ if } \frac{dm_{ij}(t)}{dt} \geq 0, \\
\frac{dM_{ij}(t)}{dt} &= 0, \text{ if } \frac{dm_{ij}(t)}{dt} \leq 0,
\end{aligned} \tag{9}
$$

and will therefore be determined, except for an arbitrary initial value, for all values of t. If we take this value as zero at some specific time t_0, we shall have the formula

$$
M_{ij}(t) = \frac{1}{2}\int_{t_0}^{t} \left\{ \frac{dm_{ij}(t)}{dt} + \left|\frac{dm_{ij}(t)}{dt}\right| \right\} dt. \tag{9.1}
$$

The function $M_{ij}(t)$ is thus seen to be not merely positive, but also non-decreasing; that is to say, $M_{ij}(t_2) \geq M_{ij}(t_1)$ if $t_2 > t_1$. It represents the total of credits which have been extended by the market (i) to the market (j) in the interval of time (t_0, t), regardless of the credits which may have been subtracted during this time by other transactions which may have transpired.

Exercise.—Let $m_{12}(t)$ increase at the rate of one million dollars a week for the first three weeks after $t = t_0$, then remain constant for a week, then decrease at the rate of half a million dollars per week for the next two weeks, and finally increase for the next three weeks at the rate of half a million dollars per week. Take the unit of time as the week, and draw graphs of $m_{12}(t)$ and $M_{12}(t)$ from $t = t_0$ to $t = t_0 + 9$.

Consider now the transactions in the various markets which consist of the sale of goods for which bills of exchange are given. As a typical instance let an amount of goods α be sold in the market (2) at a price p, and let the resulting money or check or bill of exchange be deposited in the market (1). This bill may consist of credit in the dollars of (2) so that there subsists the equation

$$\Delta m_{12} = \Delta M_{12} = c_{21} p \alpha$$

On the other hand, the credit may consist in the right to dollars of some other market (k), so that there results the equation

$$\Delta m_{1k} = \Delta M_{1k} = c_{21} p \alpha$$

If the credit consists of a check on the bank at (1) from a bank at (2) or (k) we shall still have the same forms of equation.

In general if there are deposited in unit time in the market (1) bills on markets (2), (3), . . . (n) for goods of various amounts u_{ij}, v_{ij} w_{ij}, . . . sold in the various markets (j) at prices $p_j^{(u)}$, $p_j^{(v)}$, $p_j^{(w)}$, . . . , the above equation yields, by definition of M_{ij} the equation

$$\begin{aligned} c_{21}(p_2^{(u)}u_{12} + p_2^{(v)}v_{12} + p_2^{(w)}w_{12} + \cdots) \\ + c_{31}(p_3^{(u)}u_{13} + p_3^{(v)}v_{13} + \cdots) \\ + c_{n1}(p_n^{(u)}u_{1n} + p_n^{(v)}v_{1n} + \cdots) = \frac{dM_{12}}{dt} + \frac{dM_{13}}{dt} + \cdots \\ + \frac{dM_{1n}}{dt}. \end{aligned} \qquad (10)$$

Of course there may occur transactions in which goods are sold in the market (1) and the proceeds deposited in (1); such transactions in themselves are neglected, but if they involve ultimately a transfer of exchange, that exchange may itself be considered as a commodity, and the transfer as a new transaction.

Similarly if we consider the goods sold, of which the proceeds are deposited to the credit of banks of markets other than (1) in their accounts with (1), we have the equations

Total value of such goods sold, in dollars of (1)

$$= c_{21}\frac{dM_{21}}{dt} + c_{31}\frac{dM_{31}}{dt} + \cdots + c_{n1}\frac{dM_{n1}}{dt} \qquad (11)$$

The equations (10) and (11) are both given in terms of credit at the market (1).

. Let us now denote by $T_1(t_0, t)$ the total value of trade issuing from (1) in the sense of (10), in the time t_0 to t, and by $S_1(t_0, t)$ the total value of trade entering (1), in the sense of (11), during the same interval of time, both values being given in terms of dollars of (1). The quantities $T_1(t_0, t)$ and $S_1(t_0, t)$ are merely the integrals of the members of (10) and (11), respectively, from t_0 to t. The difference

$$B_1(t_0, t) = T_1(t_0, t) - S_1(t_0, t) \qquad (12)$$

may be called the *balance of trade* in favor of the market (1).

The corresponding quantities $T_i(t_0, t)$, $S_i(t_0, t)$ may be defined for each of the other markets, and the equations analogous to (10) and (11) may be stated in the following form:

$$T_i(t_0, t) = \sum_1^n{}'_k \left\{ M_{ik}(t) - M_{ik}(t_0) \right\} \qquad (13)$$

$$S_i(t_0, t) = \int_{t_0}^{t_1} \left\{ \sum_1^n{}'_k c_{ki}\frac{dM_{ki}}{dt} \right\} dt, \quad i = 1, 2, \ldots, n,$$

where, as usual, $\sum_1^n{}'_k$ means the sum of all the terms of the kind specified with the exception of those for which $k = i$.

47. Theory of Approximate Rates of Exchange.—Various theoretical relations between the rates of exchange c_{ij} can now be obtained, by making hypotheses about the relations between $T_i(t_0, t)$ and $S_i(t_0, t)$. If we assume that the only economic communication between the markets is of the kind described, even the transport of money being considered as a transaction of sale for exchange, the credits of one market can only be met by the credits in the hands of other markets, and continued sales will occur only if these accounts remain approximately balanced, or, periodically, become approximately balanced. Here, the measure of approximation should refer to the total volume of trade, since a comparatively large balance against a particular market will be assumed not to affect greatly the rate

of exchange, if there is the prospect that a large trade will take care of it in the future. It is for this reason that the quantities $M_{ij}(t)$ were introduced, and the actual balances $m_{ij}(t)$ were not considered as sufficient indices of the situation.

If we make the hypothesis that credits for and against each market remain approximately balanced, we have as an approximation the equations

$$T_i(t_0, t) = S_i(t_0, t), \; i = 1, 2, \; . \; . \; . \; , n, \text{ if } t - t_0 \text{ is large,} \qquad (14)$$

and on the basis of the hypothesis that the credits are periodically balanced after intervals of time $t_1 - t_0 = T$, we have the approximate equations

$$T_i(t_0, t_1) = S_i(t_0, t_1), \; i = 1, 2, \; . \; . \; . \; n. \qquad (15)$$

It is natural to take $t_1 - t_0$ as one year, on account of the seasonal phenomena connected with trade; but shorter or longer intervals may also be considered.

Let us consider the equations (15), and obtain an approximate theory by assuming that the c_{ij} are constants during the interval of time from t_0 to t_1

$$c_{ij}(t) = C_{ij} \qquad (16)$$

Since the $c_{ij}(t)$ are now constants, they may be taken outside the integral sign in (13), and the equations (15) may be written in the form

$$\sum_{1}^{n}{}_j' \{M_{ij}(t_1) - M_{ij}(t_0)\} = \sum_{1}^{n}{}_k' C_{ki}\{M_{ki}(t_1) - M_{ki}(t_0)\}. \qquad (16.1)$$

Here without loss of generality we may assume that the $M_{ij}(t_0)$ are all zero, since that supposition is equivalent merely to designating the brackets in the above equation by other symbols.

If the situation is examined where there are the three markets (1), (2), (3), only, the equations (15) become the following:

$$\begin{aligned} M_{12} + M_{13} &= C_{21}M_{21} + C_{31}M_{31} \\ M_{21} + M_{23} &= C_{12}M_{12} + C_{32}M_{32} \\ M_{31} + M_{32} &= C_{13}M_{13} + C_{23}M_{23} \end{aligned} \qquad (17)$$

and here, on account of the relations (2), which we may use in the form

$$C_{ij} = \frac{C_{i1}}{C_{j1}}$$

these equations involve only the two rates C_{21} and C_{31}. We have

$$\begin{aligned} M_{21}C_{21} + M_{31}C_{31} \qquad\qquad &= M_{12} + M_{13} \\ (M_{21} + M_{23})C_{21} - M_{32}C_{31} \qquad\qquad &= M_{12} \\ -M_{23}C_{21} + (M_{31} + M_{32})C_{31} &= M_{13} \end{aligned} \tag{18}$$

The equations (18) constitute three equations in only two unknowns, C_{21} and C_{31}, but they are consistent, for if we subtract the second from the first we get the third. In other words, if we can solve two of the equations for C_{21}, C_{31}, the same values will also satisfy the third. The vanishing of the determinant of the coefficients here takes place identically, unlike the case of (7), and therefore yields no relation among the M_{ij} which has to be satisfied.

Since the three equations (18) say no more than any two of them, we can solve any two of them as linear equations and determine the unknowns C_{21}, C_{31}

$$\begin{aligned} C_{21} &= \frac{M_{12}M_{32} + M_{12}M_{31} + M_{13}M_{32}}{M_{21}M_{32} + M_{21}M_{31} + M_{31}M_{23}} \\ C_{31} &= \frac{M_{12}M_{23} + M_{21}M_{13} + M_{13}M_{23}}{M_{21}M_{32} + M_{21}M_{31} + M_{31}M_{23}}, \end{aligned} \tag{19}$$

this, of course, with the proviso that the denominator does not vanish.

48. Separation of Markets.—The quantities c_{21}, c_{31} will fail to be given by the formulae (19) if the denominator in (19) happens to vanish, that is, if

$$M_{21}M_{32} + M_{21}M_{31} + M_{31}M_{23} = 0.$$

Since however the quantities M_{ij} are all $\geqq 0$, this will be the case only if the three terms of this expression vanish separately; *i.e.*, if

$$M_{21}M_{32} = 0,\ M_{21}M_{31} = 0,\ M_{31}M_{23} = 0.$$

In order to satisfy these three equations at the same time, one of the following situations must arise

$$M_{21} = 0,\ M_{31} = 0 \tag{a}$$
$$M_{21} = 0,\ M_{23} = 0 \tag{b}$$
$$M_{32} = 0,\ M_{31} = 0. \tag{c}$$

In order to satisfy these conditions, it is not necessary that the C_{ij} be infinite or zero. This is seen clearly by returning to the equations (17). Consider the case (a). The first of the equa-

tions (17), by means of (*a*), reduces to $M_{12} + M_{13} = 0$, whence $M_{12} = 0$ and $M_{13} = 0$. Hence the second of (17) reduces to

$$M_{23} = C_{23}M_{32}$$

and the third to

$$M_{32} = C_{23}M_{23},$$

both of which relations state the same thing. Accordingly in the case (*a*) we have

$$M_{12} = M_{21} = M_{13} = M_{31} = 0 \qquad (a')$$
$$C_{23} = \frac{1}{C_{32}} = \frac{M_{32}}{M_{23}}.$$

This situation may be described by saying that the first market has no trade with either of the others, so that the rates of exchange involving it are indeterminate; and the rates of exchange of (2) with (3) are given directly by equations involving those markets alone.

EXERCISE.—Show that in the case (*b*) the market (2) drops out, and in the case (*c*), the market (3).

49. Further Theories of Exchange.—We have obtained a sort of first approximation for the values of the rates of exchange by finding the constant solutions of (15). On the other hand, if we assumed the equations (14) to be true all the time, we should have the $c_{ij}(t)$ solutions of the equations obtained by differentiating (14), *viz.*,

$$\sum_{k=1}^{n}{}' \frac{dM_{ik}}{dt} = \sum_{k=1}^{n}{}' c_{ki} \frac{dM_{ki}}{dt}, \; i = 1, 2, \cdots, n. \qquad (20)$$

A further approximation to the theory of exchange could be a compromise between the values obtained by (15) and those obtained by (20). An emphasis on (20) is an emphasis on the momentary character of the market, whereas a special weight given to (15) is an emphasis on the average character of the market.

The equations (14), (15) do not exhibit in any obvious fashion their similarity to the earlier equations involving prices and quantities brought to market in unit time. The quantities $c_{ij}(t)$ are however readily seen to be the prices of rights to credit at time *t*, and the equations (14) and (15) may be interpreted as obtained by equating offer and demand. In fact, the exchange is sold and bought competitively so that the analysis of strict

competition may be regarded as applicable, but the number of kinds of rights, each for a different combination of markets and with a different price, cannot be regarded as a single commodity, and the equations become simultaneous equations. In this sense the quantities $S_i(t_0, t)$ may be regarded as demands, in the market (i), and the quantities $T_i(t_0, t)$ as offers, their derivatives as demands and offers per unit time, and the equations (14) or (15) as stating the conditions of equilibrium for the determination of prices by writing offer equal to demand.

Various generalizations of the theory here presented will be suggested to the reader by the analysis of Chapter IV.

50. General Exercises.

1. Obtain in the case of four markets the theory of the separation of markets analogous to that of Sec. **48.** The reader will find, on the basis of a discussion of determinants, a treatment of this theory of n markets by H. E. BRAY, *Amer. Math. Monthly*, Nov.-Dec., 1922.

2. What is the effect of the cost of transportation of gold on the rates of exchange, if as much gold is available as desirable? Th reader may consult on this subject the treatment in the translation of COURNOT's "Mathematical Principles of the Theory of Wealth," New York, 1927.

3. By means of a redefinition of the symbols $M_{ij}(t)$, or with the introduction of new symbols, treat the more general situation where there will be more than one bank in each market.

CHAPTER VIII

THE THEORY OF INTEREST

51. An Elementary Theory.—One portion of economic discussion relates to the subjects of capital and interest, a discussion which has been much simplified by means of what is called the "mathematics of finance." Indeed, this simple branch of mathematics has made as extensive a contribution to that part of economic theory as the mathematical analysis of statistics has made to the inductive part which deals with the correlation of data. Hence it is the purpose of this chapter not so much to develop the technique by means of which annuities are reckoned and incomes capitalized, that being already the subject of the above analysis, but rather to show how things like interest fit into the rest of our economic theory.

The essential characteristic of the transfer of credit or income in time, as distinguished from the transfer of credit in space, is that the demand for the transfer is essentially one sided, so that there is a premium on present over future income. Although the owner of an oil well may be willing to pay a slight fee for the sake of preserving his future income in a safer place than his stocking or some secret receptacle in the chimney, he is nevertheless already the possessor of income in the future merely by diverting some of his present income. On the other hand the owner of a growing alligator-pear tree or pecan grove must actually buy present income, until his trees mature. If for every dollar of present income that he wishes to buy he has to promise $1 + \mu$ dollars of future income at a time T later, μ will be called the rate of interest for the period T. Usually a percentage rate of interest is given, which is the premium on a hundred dollars, instead of one dollar.

In order to get a simple theory we shall first consider merely incomes at a present time t, and at a single future time $t + T$. Suppose that there are n individuals in the community with greater or less preferences for present over future incomes, and that they possess incomes which, unmodified by borrowing, would be $c_1', c_2', \ldots c_n'$ at the time t, and $c_1'', c_2'', \ldots c_n''$ at

time $t + T$. Suppose that their incomes are changed by respective borrowings and lendings to

$$x_i' = c_i' + \xi_i', \text{ at time } t,$$

and

$$x_i'' = c_i'' + \xi_i'', \text{ at time } t + T.$$

The amount ξ_i' which each individual would borrow at time t, considering lending as negative borrowing, would be given, in terms of the interest he would be willing to pay, by means of a demand function

$$\xi_i' = \varphi_i(c_i', c_i'', \mu_i) \tag{1}$$

μ_i being the arbitrary or hypothetical rate of interest. This equation (1) is nothing else but the sort of demand or offer law we have been dealing with in previous theories, where now ξ_i' corresponds to u_i and μ_i to p. If we assume that these equations may be solved for the quantities μ_i, we shall have relations of the form

$$\mu_i = F_i(c_i', c_i'', \xi_i'), i = 1, 2, \cdots, n. \tag{2}$$

It may occur to the reader to substitute for the equations (2) the apparently more general equations

$$\mu_i = F_i(c_i', c_i'', \xi_i', \xi_i''), i = 1, 2, \cdots, n,$$

which if solved for the ξ_i' would yield equations of the form

$$\xi_i' = \varphi_i(c_i', c_i'', \xi_i'', \mu_i).$$

But in these individual demand and offer functions the ξ_i'' are not the actual final values of those quantities after the time T, but, from the lenders' or borrowers' points of view, are merely the quantities $-\xi_i'(1 + \mu_i)$. In other words, the F_i do not represent the individuals' points of view if they involve quantities which cannot be known until after the interval of time T, and otherwise, these functions can be simplified into the forms given in (2) if the ξ_i'' are the quantities $-\xi_i'(1 + \mu_i)$. The basic assumption underlying equations (1) and (2) is that the situation is one where there is an individual offer or demand; presumably then, that the operation of lending is competitive, and competitive in the strict sense.

In addition to the equations (1) or (2) given above, if there is to be a single rate of interest μ, we have

$$\mu_1 = \mu_2 = \cdots = \mu_n = \mu, \tag{3}$$

and if all loans are to be repaid after time T, the n equations

$$\xi_i'(1 + \mu) + \xi_i'' = 0 \qquad i = 1, 2, \cdots n, \tag{4}$$

which give the future values ξ_i'' of present changes of income ξ_i' in terms of μ, according to the definition of rate of interest, and correspond to the equations (3) in the theory of rates of exchange. We must also write the equations

$$\xi_1' + \xi_2' + \cdots + \xi_n' = 0 \tag{5}$$

$$\xi_1'' + \xi_2'' + \cdots + \xi_n'' = 0 \tag{5.1}$$

which balance accounts at the times t and $t + T$ respectively. Equation (5.1) may be deduced directly from (4) and (5), and is therefore redundant; in fact $\Sigma\xi_i'' = -(1 + \mu)\Sigma\xi_i'$.

The unknowns $\xi_1'\xi_2', \cdots \xi_n', \xi_1''\xi_2'', \cdots \xi_n''$ are now to be determined from the equations (2), (3), (4), (5), or (1), (3), (4), (5). The auxiliary variables μ_i are at once eliminated, and we have the equations

$$\xi_i' = \varphi_i(c_i', c_i'', \mu) \tag{6}$$

to determine $\mu_1, \xi_1', \ldots, \xi_n'$. Hence we have, to determine μ, the equation

$$\sum_1^n {}_i\varphi_i(c_i', c_i'', \mu) = 0 \tag{7}$$

an equation which is solvable for μ if

$$\sum_1^n \frac{\partial\varphi_i}{\partial\mu} \neq 0, \tag{7.1}$$

that is to say, unless the μ has been eliminated, in adding the equations (6) together, by some accidental character of the functions φ_i, in such a way that (7) does not involve μ at all.

Having determined μ from (7) the ξ_i' are determined from (6) and the ξ_i'' from (4), so that the problem is completely solved.

52. Special Cases.—One or two simple illustrations will help to clarify these relations. Let us take as demand equations the simplest formulae possible, writing them as linear equations of the kind considered in previous chapters:

$$\xi_i' = a_i\mu_i + b_i, \quad b_i > 0,\ a_i < 0, \tag{8}$$

the a_i, b_i being functions of the known quantities c_i', c_i''.

$$a_i = a_i(c_i', c_i''), \quad b_i = b_i(c_i', c_i'').$$

Then equation (7) becomes

$$\left(\sum_1^n a_k\right)\mu + \sum_1^n b_k = 0$$

whence

$$\mu = -\frac{\Sigma b_k}{\Sigma a_k} \tag{9}$$

unless $\Sigma a_k = 0$. The exceptional case when $\Sigma a_k = 0$ is that where μ is undetermined by the given assumptions. Here $\Sigma \partial\varphi_i/\partial\mu = 0$. If Σb_k is also zero any value of μ will satisfy the equations, but otherwise, in this case, the hypotheses are contradictory.

Finally

$$\begin{aligned} \xi_i' &= -a_i\frac{\Sigma b_k}{\Sigma a_k} + b\cdot = \frac{b_i\Sigma a_k - a_i\Sigma b_k}{\Sigma a_k}, \\ \xi_i'' &= -(1+\mu)\xi_i' = \frac{(\Sigma b_k - \Sigma a_k)(b_i\Sigma a_k - a_i\Sigma b_k)}{(\Sigma a_k)^2} \end{aligned} \tag{10}$$

In particular if the ratio b_i/a_i is independent of i, that is, if

$$a_i = af_i(c_i', c_i''), b_i = bf_i(c_i', c_i''), \tag{11}$$

equations (9) yield the value

$$\mu = -\frac{b}{a}; \tag{11.1}$$

and the equations (10) yield the values

$$\xi_i' = 0, \xi_i'' = 0, i = 1, 2, \cdots, n.$$

In other words, the equilibrium value of μ is perfectly determinate but has such a value that no borrowing or lending takes place, whatever the initial values c_i', c_i''. Conversely, from equations (10), if no borrowing or lending takes place we must have

$$b_i\Sigma a_k = a_i\Sigma b_k = 0, i = 1, 2, \cdots, n$$

or

$$\frac{b_i}{a_i} = \frac{\Sigma b_k}{\Sigma a_k}$$

which states that the ratio b_i/a_i is independent of i.

If we particularize otherwise the equations (8) by writing

$$a_i = a, b_i = b\frac{c_i''}{c_i'},$$

which is an obviously practical hypothesis, the demand and offer equations become the following:

$$\xi_i' = a\mu_i + b\frac{c_i''}{c_i'},\ a_i = a,\ b_i = b\frac{c_i''}{c_i'} \tag{12}$$

The value of μ becomes

$$\mu = -\frac{b}{a}\frac{1}{n}\sum\frac{c_k''}{c_k'} \tag{13}$$

thus given in terms of averages of the ratios c_k''/c_k'. Also, from (12),

$$\xi_i' = b\left\{\frac{c_i''}{c_i'} - \frac{1}{n}\sum\frac{c_k''}{c_k'}\right\},$$

so that ξ_i' is positive or negative according as the ratio c_i''/c_i' for the individual exceeds or not the average value.

53. Dependence of Value on Time.—As we have seen in defining the rate of interest μ, the value of a credit which is of amount ξ at time t_0 becomes $\xi(1 + \mu)$ after the interval of time T, that is to say, with the amount ξ at time t_0 can be bought the right to the amount $\xi(1 + \mu)$, payable at the later time $t_0 + T$. If the same rate of interest continues, and the interest is compounded after each period of time T, the amount ξ paid at time t_0 will buy the right to the future credit of amount

$$A(t_0 + 2T) = \xi(1 + \mu)(1 + \mu) = \xi(1 + \mu)^2$$

at the time $t_0 + 2T$, or $\xi(1 + \mu)^3$ at time $t_0 + 3T$, or in similar fashion, after k periods T, the amount

$$A(t_0 + kT) = \xi(1 + \mu)^k,\ t = t_0 + kT.$$

On the other hand, the value at time t_0 of an amount ξ which is to be paid at time $t_0 + T$ is

$$A(t_0) = \frac{\xi}{1 + \mu},$$

and if the amount ξ is to be paid at time $t_0 + kT$, its value at t_0 is

$$A(t_0) = \frac{\xi}{(1 + \mu)^k} = \xi(1 + \mu)^{-k}.$$

Thus if the per cent rate of interest is 6 per cent per year and T = one year, we have $\mu = .06$ and the value of \$100 after 10 years $100(1.06)^{10}$; on the other hand the present value of the

right to \$100, 10 years from now is only $100/(1.06)^{10} = 100(1.06)^{-10}$. Although it is a convenience to have special tables for such quantities as occur in these problems, the theory of the construction of such tables, or of such calculations, is very simple. Since we have

$$\log a^m = m \log a,$$

whether m is positive or negative, we may write

$$\begin{aligned} \log (1.06)^{10} &= 10 \log 1.06 = 0.253 \\ \log 100(1.06)^{10} &= \log 100 + \log (1.06)^{10} = 2.000 + 0.253 = 2.253 \\ 100(1.06)^{10} &= 179 \text{ [dollars]} \end{aligned}$$

Similarly

$$\log (1.06)^{-10} = -10 \log 1.06 = -0.253$$

the last number being often written as $0.747 - 1 = \bar{1}.747$. Hence

$$\log 100(1.06)^{-10} = \log 100 + \log (1.06)^{-10} = 2 + (0.747 - 1) = 1.747$$

$$\frac{100}{(1.06)^{10}} = 100(1.06)^{-10} = 55.9 \text{ (dollars)}$$

If the interest rate is 6 per cent per year and the time interval T is to be taken as one month, so that interest can be calculated after one month, or several months, being compounded at each monthly period, we write $\mu = 0.06/12 = 0.005$. In this fashion we can make the period T anything we please.

EXCERCISE.—What is the value of \$100 after 10 years if the rate is 8 per cent per year? 10 per cent? If the rate is 6 per cent per year but the interest is compounded monthly?

One of the most frequent problems in this sort of value is that of the determination of the value at a specific time of a series of payments. For example, what amount at the present time could be regarded as equivalent to 96 payments, at intervals of one month, of \$25 each, the first payment being made at the present time, and the rate of interest being 6 per cent per year, compounded monthly?

In general, what is the value at time t_0 of a series of n payments of amount ξ, the interval between payments being T and the first payment taking place at time t_0? When we know the value A at time t_0 we can of course write down the value at any other interest period $t_0 + mT$ as $A(t_0 + mT) = A(t_0)(1 + \mu)^m$.

If the rate of interest is taken as μ for the time T, and interest is compounded every period, the value at time t_0 of the $(k + 1)^{st}$ payment is

$$\frac{\xi}{(1 + \mu)^k},$$

for the first payment is made at time t_0 and the $(k + 1)^{st}$ is accordingly made after k periods T. Hence the total value at the time t_0 of the n payments will be

$$\begin{aligned} A(t_0) &= \xi + \frac{\xi}{1 + \mu} + \frac{\xi}{(1 + \mu)^2} + \cdots + \frac{\xi}{(1 + \mu)^{n-1}} \\ &= \xi\left(1 + \frac{1}{1 + \mu} + \frac{1}{(1 + \mu)^2} + \cdots + \frac{1}{(1 + \mu)^{n-1}}\right) \\ &= \xi(1 + r + r^2 + \cdots + r^{n-1}), \end{aligned}$$

if we write

$$r = \frac{1}{1 + \mu}.$$

We have then the formula

$$A(t_0) = \xi\frac{1 - r^n}{1 - r}, \quad r = \frac{1}{1 + \mu},$$

which may also be written in the form

$$A(t_0) = \frac{1 + \mu}{\mu}\xi\left\{1 - \frac{1}{(1 + \mu)^n}\right\}.$$

In the specific case mentioned $n = 96$, $\xi = 25$, $\mu = 0.005$ and therefore

$$A(t_0) = \frac{1.005}{0.005}25\left\{1 - \frac{1}{(1.005)^{96}}\right\}.$$

Here

$$\begin{aligned} \log (1.005)^{-96} &= (-96)(0.0022) = -0.21 = 0.79 - 1 \\ (1.005)^{-96} &= 0.62 \\ 1 - \frac{1}{(1.005)^{96}} &= 0.38, \end{aligned}$$

from which the rest of the calculation may be easily made. But it is to be noticed that the usual logarithm table is unsatisfactory for practical calculations, since in this example, as a typical case, a four place logarithm table yields only two significant figures (notice the number 0.0022 above), and the

table would have to be very much extended to cover the cases apt to be required.[1]

The formulae just developed cover a multitude of practical problems. For instance, if we want to know how many monthly payments of 25 dollars, the first being made at the present time, will take care of a present debt of 1,000 dollars, the interest being compounded monthly at a rate of 6 per cent per year, we have

$$\mu = 0.005,\ r = \frac{1}{1+\mu} = 0.9950,$$

and $\log r = -0.0022 = 0.9978 - 1$. Substituting in the formula for $A(t_0)$, we have

$$1{,}000 = 25\,\frac{1 - (0.9950)^n}{0.0050}$$

which may first be solved for $(0.9950)^n$ and then, by taking the logarithm, for n.

EXERCISE.—Complete the calculations of the previous paragraphs.

54. Theory of Interest for an Arbitrary Number of Periods of Time.—We now return to our central problem. If the loans are all to be closed after h periods T instead of a single period T, the demand and offer equations, written in terms of hypothetical rates of interest would be

$$\xi_i^{(k)} = \varphi_i^{(k)}(c_i', c_i'', \cdots, c_i^{(h+1)}; \mu', \mu'', \cdots, \mu^{(h)}),\quad i = 1, 2, \cdots, n \quad k = 1, 2, \cdots, h, \tag{14}$$

where the $c_i^{(k)}$, $\xi_i^{(k)}$ denote the unmodified incomes and the borrowings respectively of the i^{th} individual at the time $t + (k-1)T$, and $\mu_i^{(k)}$ the arbitrary rate of interest which the i^{th} individual would pay for the period from $t + (k-1)T$ to $t + kT$, this interest not assumed to be constant in time, as in Sec. **53.** There is no need of writing equations (14) for $\xi_i^{(h+1)}$ since the demands are connected by the equations

$$\xi_i' + \frac{1}{1+\mu_i'}\,\xi_i'' + \frac{1}{(1+\mu_i')(1+\mu_i'')}\,\xi_i''' + \cdots + \frac{1}{(1+\mu_i')(1+\mu_i'')\cdots(1+\mu_i^{(h)})}\,\xi_i^{(h+1)} = 0,\quad i = 1, 2, \cdots, n, \tag{14.1}$$

[1] Tables for this kind of calculation are those of GLOVER, "Tables of Applied Mathematics in Finance, Insurance, Statistics," Ann Arbor, 1923.

obtained from the definition of a rate of interest, viz., that a credit of m at a time $t + (k - 1)T$ is equivalent to a credit of $m(1 + \mu^{(k)})$ at a time $t + kT$.

If we now express the hypothesis that there is a unique rate of interest for all individuals, for any period T of time, we have

$$\mu_i^{(k)} = \mu^{(k)}, \tag{15}$$

and if accounts balance at any of the times, when the lendings and borrowings of all the n individuals are considered, we have the equations

$$\xi_1^{(k)} + \xi_2^{(k)} + \cdots + \xi_n^{(k)} = 0,\ k = 1, 2, \cdots, h. \tag{16}$$

The equation (16) is of course true when $k = h + 1$, but it is a consequence of (14) and (15) and the earlier equations (16).

We have therefore the following set of equations to solve:

$$\begin{aligned} \xi_1^{(k)} + \xi_2^{(k)} + \cdots + \xi_n^{(k)} &= 0,\ (k = 1, 2, \cdots, h), \\ \xi_i^{(k)} &= \varphi_i^{(k)}(c_i', c_i'', \cdots c_i^{(h+1)}; \mu', \mu'', \cdots, \mu^{(h)}), \\ &\qquad \begin{pmatrix} k = 1, \cdots, h. \\ i = 1, 2, \cdots, n \end{pmatrix}. \end{aligned} \tag{17}$$

If then we write

$$\varphi^{(k)} = \varphi_1^{(k)} + \varphi_2^{(k)} + \cdots + \varphi_n^{(k)}$$

we have to determine the h quantities $\mu^{(k)}$ by means of the h equations

$$\varphi^{(k)}(c_i', \cdots c_i^{(h+1)}; \mu', \cdots \mu^{(h)}) = 0,\ k = 1, 2 \cdots h \tag{18}$$

It is known that these equations will be independent and will determine the solutions unless the so-called Jacobian determinant

$$\begin{vmatrix} \dfrac{\partial\varphi'}{\partial\mu'} & \cdots & \dfrac{\partial\varphi'}{\partial\mu^{(h)}} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \dfrac{\partial\varphi^{(h)}}{\partial\mu'} & \cdots & \dfrac{\partial\varphi^{(h)}}{\partial\mu^{(h)}} \end{vmatrix} \tag{18.1}$$

happens to be zero.

But when the $\mu', \cdots \mu^{(h)}$ are determined, the other unknowns $\xi_i^{(k)}$ for $i = 1, 2, \cdots, n$ and $k = 1, 2, \cdots, h$ are given at once by the demand and offer equations, and the $\xi_i^{(h+1)}$ are given be (14.1)

These results may be illustrated, as in the earlier treatments, by taking linear demand functions, which are of course accurate

enough for small ranges of values of the rates of interest. We may write, for (14), the equations

$$\xi_i^{(k)} = a_{i1}^{(k)}\mu' + a_{i2}^{(k)}\mu'' + \cdots + a_{ih}^{(k)}\mu^{(h)} + b_i^{(k)},$$
$$k = 1, 2, \cdots h,$$

and if we denote by $A_r^{(k)}$ the quantity

$$A_r^{(k)} = a_{1r}^{(k)} + a_{2r}^{(k)} + \cdots + a_{nr}^{(k)} = \sum_{i=1}^{n} a_{ir}^{(k)}$$

and by $B^{(k)}$ the quantity

$$B^{(k)} = b_1^{(k)} + b_2^{(k)} + \cdots + b_n^{(k)},$$

the equations (18) become

$$0 = A_1^{(k)}\mu' + A_2^{(k)}\mu'' + \cdots + A_h^{(k)}\mu^{(h)} + B^{(k)},$$
$$(k = 1, 2, \cdots h).$$

These have a unique set of solutions unless the determinant

$$\begin{vmatrix} A_1' & A_2' & \cdots & A_h' \\ \cdot & \cdot & \cdot & \cdot \\ A_1^{(h)} & A_2^{(h)} & \cdots & A_h^{(h)} \end{vmatrix}$$

happens to be zero. In fact this determinant is the determinant (18.1).

55. General Exercise.

1. Suppose that the rates of interest of the individuals are not the same, on account of difference in the character of the risk among them. That is to say, suppose that

$$\mu_i^{(k)} = \mu^{(k)} + \alpha_i^{(k)}$$

where the $\alpha_i^{(k)}$ are known quantities or functions of the $c_i', \cdots, c^{(h+1)}$ Determine then the quantities $\mu^{(k)}$, $\mu^{(k)} + \alpha_i^{(k)}$.

In this case, on account of the modification in (14.1), the equation

$$\xi_1^{(h+1)} + \xi_2^{(h+1)} + \cdots + \xi_n^{(h+1)} = 0$$

is not a consequence of the relations (16), and hence must be inserted as an extra equation. This implies a relation among the coefficients a_{ij}.[1]

[1] This fact was noticed by Professor Raymond Garver, in the author's course at the University of Chicago, 1925, Summer Quarter.

CHAPTER IX

THE EQUATION OF EXCHANGE

THE PRICE INDEX

56. Factors in Exchange.—Consider an economic system composed of n individuals, or parts, which for convenience we may designate as "individuals" even when they are statistical averages. Let $e_i(t)\Delta t$ be the amount of money spent by the i^{th} individual in the time from t to $t + \Delta t$, $m_i(t)$ be the average amount possessed at time t,[1] and $\alpha_i(t)\Delta t$, $\beta_i(t)\Delta t$, . . . be the amounts of various commodities α, β, . . . , purchased by the i^{th} individual in the time interval between t and $t + \Delta t$. Similarly we let $p_{\alpha i}(t)$ represent the price of unit quantity of α at time t to the i^{th} individual, since it is not essential to assume that the prices are uniform throughout the community. We wish to connect the totals and averages of these quantities in a simple manner.

In order to do this, we define the quantity $v_i(t)$

$$v_i(t) = \frac{e_i(t)}{m_i(t)} \tag{1}$$

The significance of this number in easily seen. If an individual has an income of \$2,400 a year, but receives it in payments of \$200 each month, which he spends during the same month, the average amount which he possesses will be considerably less than \$200. Hence if the year is taken as the unit of time, the $v_i(t)$ for him will be more than 12. If he gets paid by the week or the day the $v_i(t)$ will be still higher, since the average amount he possesses will be in the first case less than $1/52$ and in the second a still smaller fraction of the amount which he spends during the unit time of one year. This number is then a kind of velocity quantity, and is called the velocity of circulation, for the

[1] The average quantity $m(t)$ is the same sort of quantity as that which is used in kinetic theory, for instance to give the number of molecules in a cubic centimeter of space at time t. The quantity $m_i(t)$ is the average amount possessed by the i^{th} individual during a time Δt about t, where Δt is short enough not to hide the essential changes of the quantity, but long enough to mask the random variations.

i^{th} individual. The average of this quantity for the whole community may be described roughly as the number of times one unit of the money in circulation is used during the year. But in order to make this statement exact, the kind of average which is used must be carefully defined.

Accordingly, we extend our definitions further. Let $E(t)\Delta t$ be the total amount spent by the community from time t to time $t + \Delta t$; then

$$E(t) = \sum_{1}^{n} e_i(t). \tag{2}$$

Also let E_i be the total spent by the i^{th} individual during a time T; if we write, for convenience, $t_0 = 0$ for the beginning of the interval, we shall have

$$E_i = \int_0^T e_i(t)dt. \tag{3}$$

Again, let E be the total spent by the community in time T; then

$$E = \sum_{1}^{n} E_i = \int_0^T E(t)dt = \sum_{1}^{n}{}_i \int_0^T e_i(t)dt. \tag{4}$$

In a similar fashion we extend the definitions for $m_i(t)$. Let $M(t)$ be the total amount of money in circulation in the community at time t; we have

$$M(t) = \sum_{1}^{n}{}_i m_i(t). \tag{5}$$

Also let m_i be the average amount in possession of the i^{th} individual in the interval of time 0 to T; we define

$$m_i = \frac{1}{T}\int_0^T m_i(t)dt. \tag{6}$$

Finally, let M be the average amount in circulation in the whole community in the time from 0 to T:

$$M = \sum_{1}^{n} m_i = \frac{1}{T}\int_0^T M(t)dt = \frac{1}{T}\sum_{1}^{n}{}_i \int_0^T m_i(t)dt \tag{7}$$

Corresponding to these quantities we define also V_i, $V(t)$ and V; but we do so in such a way as to make valid the equations analogous to (1), namely

$$\frac{1}{T}E_i = m_i V_i, \ E(t) = M(t)V(t), \ \frac{1}{T}E = MV. \tag{8}$$

In order to obtain these relations the new quantities must be weighted means of the $v_i(t)$. In fact, if we substitute in (8) the values already obtained, we have

$$
\begin{aligned}
V_i &= \frac{\dfrac{1}{T}\displaystyle\int_0^T e_i(t)\,dt}{\dfrac{1}{T}\displaystyle\int_0^T m_i(t)\,dt} &&= \frac{\displaystyle\int_0^T m_i(t)v_i(t)\,dt}{\displaystyle\int_0^T m_i(t)\,dt} \\
V(t) &= \frac{\displaystyle\sum_1^n e_i(t)}{\displaystyle\sum_1^n m_i(t)} &&= \frac{\displaystyle\sum_1^n m_i(t)v_i(t)}{\displaystyle\sum_1^n m_i(t)} \\
V &= \frac{\dfrac{1}{T}\displaystyle\sum_1^n E_i}{\displaystyle\sum_1^n m_i} &&= \frac{\displaystyle\sum_1^n m_i v_i}{\displaystyle\sum_1^n m_i} = \frac{\displaystyle\int_0^T M(t)V(t)\,dt}{\displaystyle\int_0^T M(t)\,dt},
\end{aligned}
\qquad (9)
$$

$$
V = \frac{\displaystyle\sum_1^n \int_0^T m_i(t)v_i(t)\,dt}{\displaystyle\sum_1^n \int_0^T m_i(t)\,dt}
$$

The transfers of money value satisfy an equation, attributed to Simon Newcomb, which is called the equation of exchange. We have

$$e_i(t)\Delta t = p_{\alpha i}(t)\alpha_i(t)\Delta t + p_{\beta i}(t)\beta_i(t)\Delta t + \cdots$$

and, since from (8) $MV = E/T$, it follows from (4) that

$$MV = \frac{1}{T}\sum_1^n{}_i \int_0^T e_i(t)\,dt$$

And from this follows immediately the desired equation of exchange:

$$MV = \frac{1}{T}\sum_1^n{}_i\left[\int_0^T p_{\alpha i}(t)\alpha_i(t)\,dt + \int_0^T p_{\beta i}(t)\beta_i(t)\,dt + \cdots\right] \qquad (10)$$

57. Bank Deposits Subject to Check.—Up to this point we have been considering the currency as all of one kind, and we may for simplicity regard it all as money proper. In a country like the United States that means primary money—gold coins, or gold bullion—and fiduciary money—silver coin which is

redeemable in gold for more than its value as metal, and paper of various kinds redeemable in gold. In a bimetallic system there would be two kinds of primary money, based on the two metals, and fiduciary money based on each kind, or even both kinds, of primary money.

But in a modern economic system exchanges are not carried on mainly as cash transactions. Most purchases are made by check. In other words, bank deposits or credits, which often are merely deposits of loans made to the individual, for which some bank possesses a mortgage as lien on the individual's property, enable the individual to use the value of his property as a fund of property rights which he exchanges in the same way as money. That is to say, he pays by a check against his credit in the bank.

Let us reserve the letters $m_i(t)$, $v_i(t)$, M, V, etc., for amounts and velocities of circulation of money in the strict sense, and let us define analogous quantities $m_i'(t)$, $v_i'(t)$, M', V', with respect to that portion of bank deposits which are circulated through checking accounts. Irving Fisher furnishes us with the following table:[1]

	M	V	MV	M'	V'	$M'V'$
1896	0.87	19	..	2.68	36	97
1900	1.17	20	..	4.40	37	165
1905	1.45	22	..	6.54	43	282
1909	1.63	22	..	6.75	54	364

the amounts M, M' being given in billions of dollars, and the unit of time being the year.

With this extension, the equation (10) becomes the following:

$$MV + M'V' = \frac{1}{T}\sum_{1}^{n}{}_i\left[\int_0^T p_{\alpha i}(t)\alpha_i(t)dt + \int_0^T p_{\beta i}(t)\beta_i(t)dt + \cdots\right], \tag{11}$$

Both sides of this equation are of the dimension of MT^{-1}, the right hand member being the money value of total trade in unit time. We shall write this equation in the form

$$MV + M'V' = S'. \tag{12}$$

[1] "Purchasing Power of Money," pp. 280, 281, 284, 285, New York, 1922.

58. Price and Trade Indices.—It is now in order to treat the right hand member of equations (11) and (12) in much the same way as we have treated the left. Thus we may write

$$A = \sum_1^n {}_i \int_0^T \alpha_i(t)dt, \qquad B = \sum_1^n {}_i \int_0^T \beta_i(t)dt, \quad \cdot \cdot \cdot$$

$$P_A = \frac{\sum_1^n {}_i \int_0^T p_{\alpha i}(t)\alpha_i(t)dt}{A}, \quad P_B = \frac{\sum_1^n {}_i \int_0^T p_{\beta i}(t)\beta_i(t)dt}{B}$$

so that we have

$$S' = \frac{1}{T}[P_A A + P_B B + \cdot \cdot \cdot \ ]. \tag{13}$$

For convienence we consider unit time, taking $T = 1$, and we write

$$S = P_A A + P_B B \cdot \cdot \cdot = RPU, \tag{14}$$

where R is a quantity of the dimension of money value divided by time, and P and U are two dimensionless numbers, to be described.

We wish to compare money values, prices and trades through a sequence of unit times, say, for convenience, years. For this purpose we select a base year, and denote the quantities which belong to it by a subscript O. We keep R fixed and equal to S_0, and choose $P_0 = 1$, $U_0 = 1$. Hence we have

$$S = S_0 PU \tag{14.1}$$

As prices change from year to year, they may all change the same way or some may go up and others go down. If most of the prices go up, it is obvious that a wage which does not change ceases to have the same purchasing power as before; if most of the prices go down, the stationary wage increases in purchasing power. Hence it is desirable to find some ratio which will show this change in purchasing power in a quantitative way. The ratio of a weighted average or mean of prices in any given year to its value for the base year will be called a *price index* for the given year; its reciprocal will be called an *index of purchasing* power. A similar sort of ratio formed for the quantities A, B, . . . will be called a *trade index*. In (14.1), P will denote a price index and U a trade index, and the ratio S/S_0 will be called an *index of trade value*. Moreover, given P there is only one U which will with P satisfy (14.1); that value U will be called the

correlative trade index to P; and similarly P will be called the correlative price index to U.

Such averages are easily formed. We may for instance define

$$P = \frac{P_A + P_B + \cdots}{P_{0A} + P_{0B} + \cdots} \tag{15}$$

so that the correlative trade index would be

$$U = \frac{\dfrac{P_A A + P_B B + \cdots}{P_A + P_B + \cdots}}{\dfrac{P_{0A} A_0 + P_{0B} B_0 + \cdots}{P_{0A} + P_{0B} + \cdots}} \tag{15.1}$$

On the other hand, if we define

$$U = \frac{A + B + \cdots}{A_0 + B_0 + \cdots} \tag{15.2}$$

we shall have

$$P = \frac{\dfrac{AP_A + BP_B + \cdots}{A + B + \cdots}}{\dfrac{A_0 P_{0A} + B_0 P_{0B} + \cdots}{A_0 + B_0 + \cdots}} \tag{15.3}$$

The formulae thus usually go in groups of four. They may however coalesce. Thus if

$$P = \sqrt{\frac{(AP_A + BP_B + \cdots)(A_0 P_A + B_0 P_B + \cdots)}{(AP_{0A} + BP_{0B} + \cdots)(A_0 P_{0A} + B_0 P_{0B} + \cdots)}} \tag{16}$$

we get for the correlative U a formula which also results by simply interchanging the roles of the prices and quantities in (16), namely

$$U = \sqrt{\frac{(P_A A + P_B B + \cdots)(P_{0A} A + P_{0B} B + \cdots)}{(P_A A_0 + P_B B_0 + \cdots)(P_{0A} A_0 + P_{0B} B_0 + \cdots)}} \tag{16.1}$$

A more convenient set would be

$$P = \frac{AP_A + BP_B + \cdots}{AP_{0A} + BP_{0B} + \cdots} \tag{17}$$

for which

$$U = \frac{P_{0A} A + P_{0B} B + \cdots}{P_{0A} A_0 + P_{0B} B_0 + \cdots} \tag{17.1}$$

or the set

$$P = \frac{A_0 P_A + B_0 P_B + \cdots}{A_0 P_{0A} + B_0 P_{0B} + \cdots} \tag{18}$$

for which

$$U = \frac{P_A A + P_B B + \cdots}{P_{0A} A + P_{0B} B + \cdots}. \tag{19}$$

One might also use

$$P = \frac{\omega' \dfrac{P_A}{P_{0A}} + \omega'' \dfrac{P_B}{P_{0B}} + \cdots}{\omega' + \omega'' + \cdots}, \; \omega', \omega'' \text{ arbitrary consts.} \tag{20}$$

which has of course the value 1 in the base year, and therefore satisfies our definition of index number. Apparently less desirable forms are those based on geometric means, *e.g.*,

$$P = \sqrt[r]{\frac{P_A}{P_{0A}} \cdot \frac{P_B}{P_{0B}} \cdots}, \text{ for } r \text{ commodities.} \tag{21}$$

In this case, if any price becomes very small it influences abnormally the value of P since $P = 0$ when any price is zero.

59. Properties of Indices.—Not only the formulae given above, but also many others are obviously available for the definition of indices, and the question accordingly arises whether there is any theoretical ground for preference among them, or whether it is merely a practical matter of calculation or of appropriateness to a particular problem such as that of "the cost of living." On the latter ground (15) and (21) are not desirable except for prices whose ratios to base year prices remain nearly equal to unity, since prices widely different from unity might influence the price index predominantly. But otherwise, as a practical matter they all give approximately the same results. In fact even (15) is in some sense a weighted mean, since the units in terms of which prices are given correspond roughly to their use. From practical considerations the simpler formulae are better than the more complicated ones.

For theoretical use the decision between them must be made on theoretical grounds also. In order to be employed in economic laws an index must be a quantity of definite dimension, so that the statements of laws may be independent of particular units. Since however units of different commodities cannot be compared, an expression like $A + B$ is not defined as to dimension; similarly $[P_A + P_B]$ is not defined, for

$$[P_A] = \frac{[\text{Money value}]}{[A]} \text{ and } [P_B] = \frac{[\text{Money value}]}{[B]},$$

and the two terms in $P_A + P_B$ are not of the same dimension. We must therefore rule out as unfit for theoretical work such formulae as (15), (15.1), (15.2), (15.3) as not being of determinate dimension.

The quantities given by the later formulae are all of definite dimension, namely zero with respect to every fundamental unit. This in fact is the requirement that was stipulated in advance for P and U in (14) and (14.1).

On the basis of the criteria so far considered, it would appear that, of the formulae given, (16), (17), (18) give desirable definitions, for theoretical and practical work, (16) being more symmetrical but (17) and (18) simpler in calculation. It is unnecessary here to consider the various other index formulae which have been or may be defined, or to compare them with respect to other possible practical or theoretical tests. Such detailed treatment may be found in other places.[1] But it must again be emphasized that for theoretical work, homogeneity with respect to dimension is not merely desirable but essential.

Note on the Instantaneous Price Index.—The following test is one of those which Fisher has employed in order to examine possible index numbers:

The ratio of price indices for two times should agree with the price ratios for the same times, if these all agree with each other; the ratio of trade indices should agree with the trade ratios if, on the other hand, these trade ratios all agree with each other.

In particular, then, the price index should not change, whatever the changes in the various trade values A, B, $\cdots$ provided that the prices P_A, P_B, $\cdots$ do not change, and the trade index should not change in the contrary situation. By considering infinitesimal changes, F. Divisia[2] has used substantially this test in order to determine an *instantaneous* price index.

Consider for this purpose the quantities

$$A(t) = \Sigma_i \alpha_i(t), \; P_A(t) = \frac{\Sigma_i p_{\alpha i}(t)\alpha_i(t)}{\Sigma_i \alpha_i(t)},$$

with similar expressions for $B(t)$, $P_B(t)$, etc. We have

$$E(t) = A(t)P_A(t) + B(t)P_B(t) + \cdots,$$

and we may write

$$E(t) = E(t_0)P(t)U(t)$$

where $P(t)$, $U(t)$ are two numbers of dimension zero which are to be the respective instantaneous price and trade indices. In other words,

$$E(t_0)P(t)U(t) = A(t)P_A(t) + B(t)P_B(t) + \cdots \tag{22}$$

[1] The work of Irving Fisher already cited.

[2] DIVISIA, F. "Economique Rationnelle," p. 268, Paris, 1928.

This equation is valid for all possible choices of the functions $A(t)$, $P_A(t)$, $B(t)$, $P_B(t)$, $\cdots$; it implies, therefore, the relation

$$E(t_0)\{P(t)dU(t) + U(t)dP(t)\} = P_A(t)dA(t) + P_B(t)dB(t) + \cdots + A(t)dP_A(t) + B(t)dP_B(t) + \cdots \quad (22.1)$$

in the differentials of the various functions.

But the change in the price index is supposed to be determined if we know the changes in the various prices and trades, and hence $dP(t)$ is of the form

$$dP(t) = X_1dP_A(t) + X_2dP_B(t) + \cdots + Z_1dA(t) + Z_2dB(t) + \cdots,$$

where X_1, X_2, $\cdots$, Z_1, Z_2, . . . are certain unknown functions of t, depending on the $A(t)$, $P_A(t)$, etc., which we wish to determine. If now, applying the test just mentioned, $dP(t)$ is zero when $dP_A(t)$, $dP_B(t)$, $\cdots$ are all zero, we have

$$0 = Z_1dA(t) + Z_2dB(t) + \cdots$$

for all possible functions $A(t)$, $B(t)$, $\cdots$, and, that is, for all values of $dA(t)$, $dB(t)$, $\cdots$. Hence $Z_1 \equiv 0$, $Z_2 \equiv 0$, $\cdots$ and the expression for $dP(t)$ reduces to the following

$$dP(t) = X_1dP_A(t) + X_2dP_B(t) + \cdots \quad (23)$$

which consequently is valid for all values of the $dP_A(t)$, $dP_B(t)$, $\cdots$, $dA(t)$, $dB(t)$, $\cdots$.

Similarly,

$$dU(t) = Y_1dA(t) + Y_2dB(t) + \cdots \quad (23.1)$$

where the Y_1, Y_2, $\cdots$ are certain functions.

But the same test applied to (22.1) yields the equation

$$E(t_0)U(t)dP(t) = A(t)dP_A(t) + B(t)dP_B(t) + \cdots$$

when $dA(t) = 0$, $dB(t) = 0$, etc. This relation may be written in the form

$$\frac{dP(t)}{P(t)} = \frac{A(t)dP_A(t) + B(t)dP_B(t) + \cdots}{E(t)} \quad (24)$$

if we divide by (22). The formula just given holds when the $dA(t)$, $dB(t)$, . . . are all zero, but the $dP_A(t)$, $dP_B(t)$, . . . are arbitrary. Equation (23), however, is valid for all possible values of the differentials, and contains no terms in $dA(t)$, $dB(t)$, . . . Hence (24) contains no such terms even if $dA(t)$, $dB(t)$, . . . are not all zero, and the formula (24) is general.

Similarly

$$\frac{dU(t)}{U(t)} = \frac{P_A(t)dA(t) + P_B(t)dB(t) + \cdots}{E(t)} \tag{25}$$

for all choices of the functions $A(t)$, $P_A(t)$, etc.

Since for any function $f(t)$, we have $d \log f = \frac{df}{f}$, we may write (24) and (25) in the forms

$$\begin{aligned} P(t) &= e^{\int_{t_0}^{t} \frac{1}{E(t)}\{A(t)dP_A(t) + B(t)dP_B(t) + \cdots\}} \\ U(t) &= e^{\int_{t_0}^{t} \frac{1}{E(t)}\{P_A(t)dA(t) + P_B(t)dB(t) + \cdots\}} \end{aligned} \tag{26}$$

if we take $P(t_0) = 1$, $U(t_0) = 1$. But we cannot perform the indicated integrations until we know actually the various functions $A(t)$, $P_A(t)$, $\cdots$. The formulae (24), (25), (26) are the desired results, and since they satisfy (22.1), define correlative indices.

The relation of (24) to previously given formulae, such as (17) or (19), may be seen by adding unity to both sides of it. We have, from (24),

$$\frac{P(t) + dP(t)}{P(t)} = \frac{A(t)\{P_A(t) + dP_A(t)\} + B(t)\{P_B(t) + dP_B(t)\} + \cdots}{A(t)P_A(t) + B(t)P_B(t) + \cdots},$$

and if we remember that all our quantities are derived from approximations to averages, and consider these as averages during short successive intervals of time about t and $t_1 = t + dt$ respectively, taking the interval about t as the base interval, in which $P(t) = 1$, we obtain from the preceding formula the approximation

$$P(t_1) = \frac{A(t)P_A(t_1) + B(t)P_B(t_1) + \cdots}{A(t)P_A(t) + B(t)P_B(t) + \cdots}$$

This is substantially (17).

60. Purchasing Power and Its Stabilization.—We have already defined $1/P$ as an index of purchasing power, and we may regard that number as in some sense a measure of the value of the dollar. For in order to make wages have about the same

"purchasing power" in various years, they would have to be made proportional to some price index, say of the form

$$P_i = \frac{\alpha_{0i}P_A + \beta_{0i}P_B + \cdots}{\alpha_{0i}P_{0A} + \beta_{0i}P_{0B} + \cdots}$$

where the α_{0i}, β_{0i}, . . . are the quantities of the various commodities, used on the average, year by year, by the wage earner. Since as a practical result the various price indices are usually of about the same order, it would not make a great deal of difference, in comparison with the actual changes of prices, if one were used rather than another, and thus a single index adopted for the whole community.

We might propose then to guarantee wage earners against fluctuations of prices by making wages proportional to the price index. Presumably there are objections to this method. It seems a radical change, although it would not now be as radical an innovation as it would have been even a decade ago, before the price index was part of the consciousness of the community; for now it is discussed in the magazines and its value is published frequently in the daily papers. It is perhaps a cumbrous method of adjustment. Possibly it is true that wage earners would greet an apparent increase of wages, following the price index, with pleasure, but would not be as easily reconciled to an apparent diminution, even if purchasing power were unchanged. Yet in recent years, it is the fact that wage earners themselves have occasionally voted a reduction of wages for the sake of the industry and the safe-guarding of their employment. Finally, it may be said that such a method would stabilize wages, but would not control the prices of goods, or the return on other services than those paid for in wages, and therefore would not control the stability of the social situation as a whole.

Another method has been proposed and urged persistently by Irving Fisher, and as a result has been mentioned in Congress. The method suggested is to regulate the redemption value in gold, of the dollar, by fixing from time to time the weight of gold that stands for a dollar. The money in circulation would be made entirely fiduciary, which practically it is at present, and the amount of gold bullion which is redeemable by unit fiduciary money would be changed, by government regulation, as the price index started to change. In other words, the government would control its buying and selling of gold bullion (would speculate!

the conservative banker would say) by fixing its price in fiduciary money. If the price index P started to increase, a greater amount of metal would be redeemable for one dollar; and if P started to decrease the value of the fiduciary dollar in gold would be set lower. In the former case the amount of fiduciary money would decrease by redemption, and in the latter case it would expand by means of the exchange of metal for it. Incidentally the amount of gold mined in the former case would tend to decrease, and, in the latter, increase, the effect on this particular industry being the same as the present effect of high and low commodity prices and wages, which influence the cost of mining gold.

The presumption is that in this way the variation of the price index P would be checked. In fact, in the left hand member of the equation of exchange

$$MV + M'V' = R_0PU$$

the quantities V and V' are largely fixed by habit, that is, the manner in which wages are paid, whether by the month, week or day, the customary retention of certain amounts in checking accounts, and so on. Moreover M' is roughly proportional to M, as the statistical analysis has shown in the past, partly because there are legal restrictions, on the ratio of reserves in banks to deposits, which prevent M'/M from exceeding a certain value. Hence, again roughly, the whole left-hand side of the equation is proportional to M. A change in the weight of the dollar is a proportional change in the value of M.

For any change in the left-hand member, a corresponding change in the right-hand member must occur. The supposition is that the correction will occur by a change in P and that thus by adjustments in the value of the fiduciary dollar, the value of P may be held constant. The validity of the supposition will readily be accepted if the instability tends to arise from undue mining or importation of gold. But it must be pointed out that regulating the left hand member of the equation of exchange regulates the product PU, rather than P directly. Does this mean necessarily that P is regulated? Obviously, if M were reduced in order to check a threatened rise in P, the P might still increase if U decreased fast enough to counterbalance it, and such a decrease in the amount of trade might be very well what would happen in a crisis. The situation might be such that producers would decrease out-put and let sales fall off at the same time that

they increased prices. One cannot then be certain of results without a closer analysis of the economic situation as a whole than seems yet to have been given.

61. The Business Cycle.—The cyclic motion of prices as a whole, known as the business cycle, is intimately connected with the price index. This phenomenon may be briefly described.

If prices, for any cause, begin to rise, those who are borrowing money on short term notes are doing so at a trivial or even a negative rate of interest, because they are borrowing more purchasing power than the note will be worth at a later time. Thus if the price index were to start at 1 with the beginning of the year and increase to 1.25 at the end of the year, the borrower of money should pay 25 per cent interest in order merely to return the purchasing power he has borrowed. With rising prices then a producer is led to extend his business rapidly by borrowing; moreover demand seems usually to be increased with a positive rate of change of price (a hypothesis which we have already considered), there are large purchases of raw materials, more money is borrowed, a "favorable" election takes place in the government, and the "prosperity" part of the cycle continues. The interest rate gradually rises, but for some time more slowly than is enough to make a real interest charge on the borrower.

But eventually the extension reaches its limit, demand is satiated, prices do not any longer increase, interest is therefore unduly high instead of unduly low, and the reverse process begins to take place. Added to this difficulty is one that we have already noticed, that expansions of business involve increases of overhead costs that cannot later be contracted, and other changes that are not reversible. Hence depression comes rapidly, loans are called in, and cannot be met in the falling market. Failures take place, and business is stagnant. Thus the cycle is complete and ready to begin another course.

It might be that to reduce the value of M in the initial stages, while trade was beginning to boom (*i.e.*, U beginning to increase), would check the rise in P and thus make the interest rate effective. This would prevent undue borrowing and eliminate at least one element of instability from the situation.

Note on Instability.—More important than the fact that crises tend to occur after approximately equal intervals of time is the characteristic that the ascending movement of price level and production is prolonged until a critical situation is met, and that

the retrograde movement, when it once commences, is continued in a similar manner. It is interesting to be able to reproduce this kind of phenomenon in a theoretical system, by means of simple hypotheses.

Consider first a single commodity, and suppose that there is a total offer of it, as in the case of strict competition [competition (*b*)]; we may assume for simplicity that the offer is given by a linear function

$$u(t) = rp(t) + s \tag{27}$$

where r and s are constants, with $r > 0$, $s < 0$, as in Sec. **7.** We shall assume also a linear law of demand; but we shall suppose that the effect of a high price or a low price is less if the general level of prices is high or low, respectively. More precisely, we shall suppose the existence of a *lag* in that this effect depends on the level of prices at a previous time, which antedates the time t by a constant interval T, and we shall write

$$y(t) = a\frac{p(t)}{P(t-T)} + b, \quad a < 0,\, b > 0. \tag{28}$$

We take $y(t) = u(t)$.

In order to determine $p(t)$ we have accordingly the equation

$$rp(t) + s = a\frac{p(t)}{P(t-T)} + b,$$

from which we calculate the result

$$p(t) = \frac{(b-s)P(t-T)}{rP(t-T) - a} \tag{29}$$

By means of (27) we have also

$$u(t) = \frac{rbP(t-T) - as}{rP(t-T) - a} \tag{29.1}$$

If we calculate the differentials of these quantities, regarding t as variable, we have

$$\begin{aligned} dp(t) &= \frac{-a(b-s)}{[rP(t-T)-a]^2}\,dP(t-T) \\ du(t) &= rdp(t) = \frac{-ar(b-s)}{[rP(t-T)-a]^2}\,dP(t-T) \end{aligned} \tag{30}$$

In the formulae (30) the coefficients of $dP(t-T)$ are essentially positive and finite, on account of the signs of a, b, r, s. In (29), $p(t)$ is given by an expression which is essentially positive and

finite. In (30), $u(t)$ is given by an expression which cannot become infinite and which is positive in the practical case; in fact if $u(t)$ approaches zero the regime becomes impractical.

Let us suppose now, in order to have as simple a system as possible, that all the commodities satisfy similar hypotheses, and let us calculate the quantities $dP(t)$, $dU(t)$ of (24) and (25). We form for this purpose the expressions

$$\frac{\Sigma u(t)dp(t)}{\Sigma u(t)p(t)}, \quad \frac{\Sigma p(t)du(t)}{\Sigma u(t)p(t)}.$$

Without calculating the explicit values of these expressions, it is evident that the result obtained by substituting in (25), (26) gives us two relations of the form

$$d \log P(t) = F(t - T)dP(t - T), \quad d \log U(t) = G(P(t - T))dP(t - T), \qquad (31)$$

where the F, G are certain functions which are positive and do not approach zero unless all the $u(t)$ approach zero. In other words, $dP(t)$ and $dU(t)$ have the same algebraic sign as $dP(t - T)$.

In order to determine $P(t)$ as a continuous solution of (31) it is sufficient to know $P(t)$ throughout some interval of time of length T, say from t_0 to $t_0 + T$. For in this way $d \log P(t)$, and therefore, by integration $P(t)$, will be determined by (31) throughout the next interval $(t_0 + T, t_0 + 2T)$ of length T, and so on. Similarly $U(t)$ will be determined by the values of $dP(t - T)$ if, in addition, we know the single value $U(t_0 + T)$. Consequently these indices will be known for all values of t, $t > t_0 + T$.

If the price index is an increasing function during the initial interval $(t_0, t_0 + T)$, both the indices $P(t)$, $U(t)$ will remain increasing functions for all later times. We have thus the ascending movement of the economic cycle, when both prices and trade increase, and this movement will continue until the situation becomes such that the hypotheses are no longer tenable. We may call such a terminal point a crisis. Similarly, a retrograde movement which endures through an interval of length T will also be prolonged indefinitely until the productions come close to zero, or the situation becomes impractical in some other way, and the hypotheses must again be changed.

62. General Exercises.

1. Compare the development of the theory of the equation of exchange with that given in Irving Fisher's book "The Purchasing Power of Money."

2. In the formula (18) the base year is that one for which the prices and quantities are indicated by the subscript 0. If we denote by P_k the price index of the k^{th} year with respect to the year 0 as base, and by P_{jk} the price index of the k^{th} year with respect to the year j as base, is $P_{12} = P_2/P_1$?

3. If the prices the second year are all in the same proportion to the corresponding prices the first year, does the price index take on the value of this constant of proportionality? Compare (15) and (18) in this regard.

4. Discuss the formula

$$p = \frac{1}{r}\left\{\frac{p_a}{p_{0a}} + \frac{p_b}{p_{0b}} + \cdots\right\},$$

as a price index for r commodities.

5. We define the instantaneous trade value index $S(t)$ as the number

$$S(t) = U(t)P(t).$$

Show that for this number the indicated integrations may be performed, and obtain an explicit formula for it. Verify the result by means of (22).

6. Does the instantaneous price index given in (26) satisfy a test similar to that of exercise 3.

7. Show that an oscillatory movement of period $2T$, price and production moving in the same sense, is obtained if the hypothesis (28) is replaced by the following:

$$y(t) = \frac{ap(t) + b}{P(t - T)}.$$

8. What is the effect of advertising on the ascending and descending intervals of price and trade levels, according to the hypotheses of exercise 11, Sec. 8?

CHAPTER X

GENERAL CONCEPTS AND METHODS

63. Economic Theory in General.—General principles are apparent in the particular phenomena which we have studied, or at least, there are some general methods, which we can make use of in unifying those separate treatments. Nevertheless we must adopt a cautious attitude towards comprehensive theories. They do of course, in their special applications, suggest the treatment of particular problems, as well as classify them. Yet this comprehensive character, which they may have as sorts of inductive syntheses of previously studied situations, may precisely in that way circumscribe our ideas, and prevent from entering our minds the observation of other classes of phenomena. We may thus consider only one part of our subject, while we are under the impression that our study is general.

In the earlier chapters, for instance, we made a fairly extensive study of particular problems by imposing hypotheses about profits. Shall we therefore infer that every situation in economics is determined by the maximizing of some corresponding collection of profits? But we should have difficulty perhaps in assimilating the problems of Chapter IV in such a scheme, as far as that scheme is suggested by the earlier treatment, for the behaviour of the profit would have to depend on the quantity dp/dt, and accordingly could not easily be determined as a maximum for any particular time. In fact, if we write, as in the monopoly problem,

$$\begin{aligned} \pi &= pu - Au^2 - Bu - C \\ u &= y \end{aligned}$$

but assume that y involves dp/dt, *viz.*,

$$y = ap + b + h\frac{dp}{dt}$$

then the π becomes a function of t, $\pi(t)$. Can we determine p as a function $p(t)$ of t, in such a way as to make $\pi(t)$ a maximum? Obviously the method of Chapter I will no longer apply. It turns out, as we shall see in Chapter XIV, that there is a way of

interpreting this problem as a question of maxima, but different methods and new concepts will be necessary for that purpose.

It is a temptation, also, to generalize a particular set of relations which has been found useful, by substituting variables for all the constants in the equations, and then, wherever there is one unknown quantity, to put in for it an arbitrary number of such quantities, much as we have done in passing from problems involving two producers to those involving n. Thus we hope to achieve generality by creating complication. And yet it may be questioned as to whether we have added to anything but our mathematical difficultities. Unless these difficulties must be surmounted for the sake of the practical application, or unless they are interesting mathematically for their own sakes, and introduce unexpected elements into the problem, we cannot say that we have gained much.

Shall we, then, with apologies to Newton and Einstein, abandon the search for general theories? Evidently such a renunciation would be extreme in the other direction, for the practical advantages are too many and the urges of a natural curiosity are too strong for us to resist them. But we shall gain much if we can formulate our propositions in such a way as to make evident the limitations of the theory itself, and thus indirectly suggest the devising of other formulations.

In other words, we must regard, as an essential part of the theory, the clarification of the assumptions on which it rests. Our aim therefore will not be to discover general natural laws in economics by a process of Baconian induction, and thus to build up a picture of the whole of society on its economic side. Useful as this program may promise to be, there have already been so many attempts to carry it out that ours may well be spared; and, indeed, the result of these efforts has not been such as to warrant much expectation of success, for it has even convinced some economists that there are no general laws in the subject. Let us admit that the entire economic aspect of human affairs is necessarily too vast to be covered by a single theory. Our endeavor then should be to make systematic discussions of several groups of economic situations, as theoretical investigations, and bring out the respective hypotheses which separate these groups.

It is unnecessary to state that the bases of action are various. Sometimes there is an attempt to unify them by saying that a

man tries to act in such a way as to increase his pleasure. But from this point of view we have to consider at the same time not only both capitalist and laborer, but also the profiteer and the soldier, the adventurer and the hermit, the teacher, the beggar and the thief. And if the bases of economic action in all these cases are described as pleasure, the term loses its meaning; for some want power, some profit, and some take part in material affairs merely because they have to. It is a question of material affairs that interests us here. And evidently the sentiments which urge or fail to urge men in the economic aspects of their lives are of all types.

Luckily we need not consider it our province to investigate all these springs of action. To say this, does not mean that such a study is not interesting. But we may split off all of these questions from our field of investigation as economists and leave them to such time as we may be sociologically inclined. Our endeavor is to analyse possible theoretical relations involving capitals, services, commodities, the creation and transfer of wealth, the statics and dynamics of manufacture and sale, considering these things as far as possible in themselves, apart from individual and mass psychology. We may make hypotheses which presuppose that there is in the practical applications of the theories something which furnishes the particular kind of motive power which is assumed, but the psychology, as psychology, need not bother us now.

Likewise, we do not have to decide which of all possible practical and theoretical worlds is best for mankind. Here again we are leaving aside questions which, as students of ethics or lovers of mankind, interest us all. And if we have found that certain forms of organization, for instance, will discourage the profiteer, and that certain forms of taxation will have their incidence heavily on those who can pay them only by lowering what is already a low standard of living, there is nothing to prevent our applying the results of our theories to practice. Public finance need not be merely a study of how to get money out of men in such a way that they shall know as little as possible of when it goes.

We may in fact make our statement more precise, and say that our principal desire in constructing economic theory is to find possibilities of better living. On this ground we may still assert that these questions are not a part of economic theory, and more-

over that the only way to know how sentiment may profitably play a role in human affairs is to know what theoretical relations there may be which are independent of sentiment. Here again sociology will be fruitful only with reference to the possible applicability of theoretical economics. Yet the latter subject is not concerned with the balance between good and evil.

64. The Mathematical Method.—The systematization which occurs in a theoretical science, as we may properly call it in order to distinguish it from a natural or an applied science, is a process which is apt to come late in the development of a subject. Evidently some fields of knowledge are hardly ready for it, for it is typified by a free spirit of making hypotheses and definitions rather than a mere recognition of facts. But when we find this feeling for hypothesis and definition and, in addition, become involved in chains of deductive reasoning, we are driven to a characteristic method of construction and analysis which we may call the mathematical method.

It is not a question as to whether mathematics is desirable or not in such a subject. We are in fact forced to adopt the mathematical method as a condition of further progress.

The first stage of the mathematical method may be regarded as synthetical. Fundamental characteristics of an object are seized upon and by their means further properties are "discovered" by reconstructing an ideal scheme of the object. Thus a straight line is recognized as indefinitely divisible, and this division is carried as far as possible, first by points that are discretely spaced, then by rational points, corresponding to fractions, which come as close together as may be desired, and finally by irrational points which separate classes of rational points. Irrational points are "discovered" on the line. For instance, the point $\sqrt{2}$ is seen as the point which separates the rational points like $3/2$, for which the square is greater than 2, from those like $4/3$, for which the square is less than 2. Evidently $\sqrt{2}$ is not a rational fractional at all. In a similar fashion other irrational points are found. In this way the concept of the infinitely divisible straight line is developed until it comes to be replaced by an ideal, ordered "continuum" of points—something which is sufficiently different from the rude characteristics which gave it birth to cause some wonder on the part of the uninitiated.

The wonder is not decreased when imaginary and complex points are inserted on the same straight line, which seems already

full enough. But where otherwise are all the solutions of algebraic equations? If we take the circles

$$(x - 2)^2 + y^2 = 1 \text{ or } x^2 - 4x + y^2 + 3 = 0 \qquad (1)$$
$$(x + 1)^2 + y^2 = 1 \text{ or } x^2 + 2x + y^2 = 0 \qquad (2)$$

which are unit circles with centers on the x-axis at points (2, 0) and (−1, 0) respectively, they both pass through the points

$$(\tfrac{1}{2}, \tfrac{1}{2}\sqrt{-5}) \text{ and } (\tfrac{1}{2}, -\tfrac{1}{2}\sqrt{-5})$$

since the values $x = \frac{1}{2}$, $y = \pm\frac{1}{2}\sqrt{-5}$ evidently satisfy both equations. This itself is perhaps strange enough. But the coordinates of any point which satisfy both of these equations

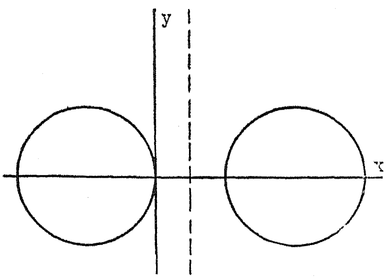

FIG. 22.

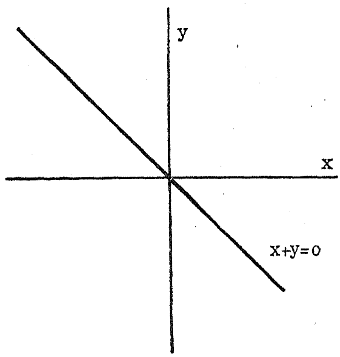

FIG. 23.

will satisfy any equation which may be obtained by combining them, and if we subtract (1) from (2) we obtain $x = \frac{1}{2}$. Hence the imaginary points of intersection of the circles lie on the vertical straight line $x = \frac{1}{2}$. To put such a difficulty even more briefly, the point $(\sqrt{-1}, -\sqrt{-1})$ lies on the line $x + y = 0$, since the coordinates $x = \sqrt{-1}$, $y = -\sqrt{-1}$ satisfy the equation of the line.

It is plain that when things get to such a state further analysis is necessary. That the difficulties in this particular case are not trivial is evidenced by the long history of the subject from the time of Pythagoras who, it is said, wished to keep irrational points a secret, through Euclid, and down to the present time. The solution of the puzzle is obtained by separating clearly the deductive scheme from its applications to any particular space, by presenting a complete array of the hypotheses or postulates which are satisfied by the elements of the system, these being

given whatever names may seem appropriate, and by devising analytical proofs of the properties of the system in terms of a strict deduction from these postulates. Thus the way is left open to substitute different hypotheses or postulates and obtain new properties. This is the second stage of the mathematical method.

Luckily the subject of economics in its present stage does not seem to offer as complicated a conceptual riddle as that of the relation of abstract geometry to "real" space; and so there is not a present need of forming a complete system of postulates for the subject. Nevertheless we do well to bring out characteristic hypotheses upon which theories rest, and thus to indicate to what extent these theories are specialized.

CHAPTER XI

UTILITY

65. Measure of Utility.—One of the general methods of theoretical economics is the discussion of economic situations with respect to their desirability. This has necessitated the construction of an artificial function, used as a measure of desirability, sometimes called utility (Jevons) or ophelimity (Pareto), or desirability (Walras). But in accepting these theories as general we encounter some of the difficulties which have been mentioned in the last chapter, and there is in some of the authors who have propounded such theories rather a serious confusion of ideas.

Such authors interpret the situation in the following way. They assume that they are dealing with two aspects of the material of economics, one subjective, the other objective. On the one hand they consider that the acts which are investigated in economics are devised for the attainment of some kind of good—for the pleasure of ownership or of prospective consumption, for the sake of doing one's duty and the resultant moral satisfaction, for power, for glory, for the sake of activity itself, or for revenge. Such pleasures, satisfactions and vanities obviously are not directly measurable; they are not quantities in the sense of Chapter II. On the other hand, there are the actual quantities of commodities and money, and the money values of services and rights. The authors with whom we are concerned, however, affirm that the use of mathematics need not be confined merely to this second set of entities, but may also be applied to the order relations among the subjective "quantities." This last statement is indeed incontestable provided such order relations can be established. The properties of inequality, equality, variation, continuous or discontinuous change, and so on, may be supposed to generate these order relations.

We may, perhaps without making any existential assumptions, denote by S the non-measurable hedonistic value (pleasure or satisfaction) for a person, in quantities $x_1, x_2, \ldots, x_n$ of various commodities. Then we invent a sort of comparison

function $U(x_1, x_2, \ldots, x_n)$ which is to increase, decrease or be constant with S, that is, if one collection of quantities $x_1, \ldots, x_n$ is preferable to another, the value of $U(x_1, \ldots, x_n)$ in the first case is to be greater than in the second; it will be still more convenient if the changes of U are also greater or less according as the changes in S are greater or less. The function $U(x_1, x_2, \ldots x_n)$ may be called a utility function and may be regarded as a scale function for the hedonistic value. If it can be constructed it will not be unique, since evidently kU or $\tan U$ or many another function of U will serve as well as U, but any such function can be used as a mathematical tool to replace S.

How is such a function U to be constructed? We consider an individual (1) with amounts $(x_1, x_2, \ldots, x_n)$ of various commodities. With some assortments of small changes $dx_1, dx_2, \ldots, dx_n$, his "satisfaction" will increase and, with others, diminish; the intermediate changes where satisfaction remains constant will be given by an equation

$$X_1(x_1, \cdots, x_n)dx_1 + X_2(x_1, \cdots, x_n)dx_2 + \cdots + X_n(x_1, \cdots x_n)dx_n = 0. \quad (1)$$

The left hand member is supposed to be positive if satisfaction increases, and negative if satisfaction decreases. The utility function $U(x_1, \ldots x_n)$ is a function which satisfies the relations:

$$\begin{aligned} dU(x_1, \cdots x_n) &= 0 \\ dU(x_1, \cdots x_n) &> 0 \\ dU(x_1, \cdots x_n) &< 0 \end{aligned} \quad (2)$$

according as the left hand member of (1) is zero, positive or negative.

66. Critique of Utility.—Consider first the case where the variables $x_1, x_2, \ldots, x_n$ are replaced by two only, say x and y. $\mu(x, y)$.
The equation (1) becomes

$$X(x, y)dx + Y(x, y)dy = 0 \quad (3)$$

or

$$\frac{dy}{dx} = -\frac{X(x, y)}{Y(x, y)} \quad (3.1)$$

The interpretation of (3) or (3.1) is that there is a *direction of indifference* corresponding to each pair of values (x, y) if these are regarded as coordinates of points in a plane; in fact, the slope of this direction is the value of dy/dx given by (3.1). As we start off in this direction, that is, as we change x and y by corresponding

amounts dx and dy, satisfaction neither increases nor decreases, since (3) will be satisfied.

Now if we plot the directions (3.1) as short lines or arrows for every point (x, y) of the plane these directions or lineal elements will generate or envelop a family of curves, as in the illustrations for Chapter IV; for the equation (3.1) is nothing but a differential equation of the first order. The various curves of the family, which are called *curves of indifference*, are merely the solutions of the differential equation.

If we solve the differential equation we obtain a solution $y = f(x, c)$ which involves an arbitrary constant c, so that by giving c various values we get various curves of the family. The functions $y = f(x, c)$ give us the "general" solution of the differential equation, which we may also write in the form

$$\varphi(x, y) = c$$

if we solve for c. Then the desired function $U(x, y)$ may be obtained by writing it as a function, more or less arbitrary, of $\varphi(x, y)$, say,

$$U(xy) = \varphi(x, y) \text{ or } (\varphi(x, y))^2 \text{ or } \sin \varphi(x, y),$$

since U will be constant whenever $\varphi(x, y) = c$, and that is, along each curve of indifference. The only restriction is that we shall choose such a function that it increases always in the proper sense, that is, that it shall be greater or less if the constant c corresponds to a curve of indifference where the satisfaction is greater or less, according as we find ourselves on one curve or the other. We can in fact make the rate of increase of U as we pass from one curve to another as large or as small as we please, and thus make U satisfy various other requirements besides the principal ones.

From another point of view, we know that (3) has an *integrating factor*. There is a function $R(x, y)$ such that when (3) is multiplied through by it, it becomes the complete differential of some function $\varphi(x, y)$; that is,

$$\frac{\partial \varphi(x, y)}{\partial x} = R(x, y)X(x, y)$$

$$\frac{\partial \varphi(x, y)}{\partial y} = R(x, y)Y(x, y),$$

while the equation for the directions of indifference, or as we may also call them, *infinitesimal loci of indifference*,

$$R(x, y)X(x, y)dx + R(x, y)Y(x, y)dy = 0$$

is still the equation (3) in that the directions of indifference are still given by (3.1). The curves of indifference are accordingly the loci $\varphi(x, y) = \text{const.}$, and the function $\varphi(x, y)$ itself, or some function of it, is a utility function. We have the theorem;

If the infinitesimal loci of indifference are given in terms of two variables, there is a utility function, which is a function of those two variables.

In the case of more than two variables the situation is not so satisfactory. For three variables the equation (3) becomes

$$X(x, y, z)dx + Y(x, y, z)dy + Z(x, y, z)dz = 0 \tag{4}$$

If we can find an integrating factor $R(x, y, z)$, so that we can multiply through by it and make the resulting expression the complete differential of some function $\varphi(x, y, z)$

$$d\varphi = RXdx + RYdy + RZdz \tag{4.1}$$

then, as before, the loci of indifference will be surfaces

$$\varphi(x, y, z) = c$$

forming a one parameter family, and φ or some function of φ may be taken as the utility function U.

But in general there is no integrating factor R which will transform (4) into a complete differential (4.1). The infinitesimal loci of indifference, or planar elements, defined by (4), do not hitch up into any one parameter family of surfaces $\varphi(x, y, z) = \text{const.}$ On the other hand, if there were a utility function $U(x, y, z)$ the surfaces $U = \text{const.}$ would form precisely such a family. Hence there is in general no possible utility function. This situation is expressed by saying that the equation (4) is in general not completely integrable.

In order that there should be an integrating factor R we should have to have

$$\frac{\partial\varphi}{\partial x} = RX, \frac{\partial\varphi}{\partial y} = RY, \frac{\partial\varphi}{\partial z} = RZ$$

since

$$\partial\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz.$$

But since

$$\frac{\partial^2\varphi}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial x}\right)$$

we must have

$$\frac{\partial}{\partial x}(RY) = \frac{\partial}{\partial y}(RX),$$

and similarly

$$\frac{\partial}{\partial y}(RZ) = \frac{\partial}{\partial z}(RY), \frac{\partial}{\partial z}(RX) = \frac{\partial}{\partial x}(RZ).$$

These equations may be expanded into the following

$$R\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) + Y\frac{\partial R}{\partial x} - X\frac{\partial R}{\partial y} = 0$$

$$R\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + Z\frac{\partial R}{\partial y} - Y\frac{\partial R}{\partial z} = 0$$

$$R\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + X\frac{\partial R}{\partial z} - Z\frac{\partial R}{\partial x} = 0,$$

and if we multiply the first of these by Z, the second by X and the third by Y, and add, the terms involving the derivatives of R will cancel. The equation then reduces to the following

$$R\left\{X\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + Y\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + Z\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right)\right\} = 0,$$

and since R itself cannot be identically zero, we must have

$$X\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + Y\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + Z\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) = 0. \quad (5)$$

But this is a condition on the X, Y, Z alone, and these therefore cannot be independent if (4) is to be completely integrable. In other words, *in general there will be no utility function in the case of three variables; for such a function to exist at all it is necessary that the X, Y, Z of* (4) *satisfy the relation* (5). The situation is of course still more complicated if $n > 3$.

Is there a way out of this difficulty? Perhaps the proponents of utility functions will say that the loci of indifference $\varphi(x, y, z) =$ const. can at once be recognized in their entirety, instead of merely in infinitesimal portions (as in (4)). The statement would amount to this: given any two situations, characterized by possession of quantities $(x_1', x_2', \ldots, x_n')$, $(x_1'', x_2'', \ldots x_n'')$, respectively, the individual can tell at once (or by carrying out some imaginary process of exchange) if the two situations are equally desirable, or if not, which is preferable, and this without regard to the order in which the various exchanges are carried out. For if the utility depends on these variables alone, that is, is a function

of $(x_1, x_2, \ldots, x_n)$ it must not depend on the process by which $(x_1', x_2', \ldots, x_n')$ is changed into $(x_1'', x_2'', \ldots, x_n'')$, but merely on the final result.

But here again the situation is not general. Thus if there are three quantities in question, such as, for instance, services for building a house, money wages and commodities which the carpenter must buy for his own sustenance, would the carpenter find it just as convenient to build the whole house and then receive a lot of money and eat a big meal, as to build little by little, receive regular wages and obtain regular meals? Is not the high price of many services due to the irregular manner in which they must be exchanged for commodities?

We do not mean to say that it is not worth while to investigate the situation in which the equation (1) for the infinitesimal loci of indifference is completely integrable, or where a corresponding equation for the whole community is completely integrable. Indeed, in a theoretical science it is desirable to make just such assumptions in order to develop special theories and consequences. In the first portion of this book such theories were developed as special cases, taking as a substitute for a utility function some such thing as profit. The mistake is to assume that such theories are general and that nothing else is possible or worth investigating. The complete integrability of (1) must be introduced as a special hypothesis, in the sense of Chapter X.

67. On Value in General.—A mathematical critique similar to that just adopted is widely applicable, and is more penetrating than an analysis in terms of loose concepts where the words themselves, by their connotations, may imply theorems of existence which are untenable. Hence it is desirable to digress slightly at this point and make a remark on value in general. The concepts of beauty, truth and good are analogous to those which we have been discussing.

In every situation there is something not of the best—some ugliness, some falsity or some evil—and so the practical judgment which is to be a basis of action is not "what situation is absolutely correct?" but "which of several situations is best?" The problem involved is the comparison of two or more groups of elements of aesthetic character. By the possibility of making a judgment at all is implied the fact that between two such groups, which are not too widely separated or which are simple in the sense of containing few enough elements, one can assign greater

value to the one than to the other. In mathematical terms, then, one is confronted with the situation which relates to equation (1); if small changes $dx_1, \ldots, dx_n$ are made, one may be able to decide roughly whether the left hand member of (1) is zero, positive or negative, interpreting the fact as a zero, positive or negative change in value.

If only two elements of (1) are allowed to vary, the equation is completely integrable, and we may therefore, with practice, be able to distinguish between values of situations which differ widely, but in a few elements. Or again, if the quantities $X_1, X_2, \ldots X_n$ can be assumed to be constants, the integral of (1) takes the form

$$X_1x_1 + X_2x_2 + \cdots + X_nx_n = c,$$

and the situations may be analyzed, as far as that hypothesis may be mantained. But in general the equation (1) need not be completely integrable, and we cannot measure the situations by any absolute value function. In other words, we can devise an approximate value function as a scale for small changes of the variables, but cannot extend it beyond a merely local field unless we are willing to make some transcendental hypothesis about the existence of such a function. The existence of such a function is not deducible from (1) itself, as far as the coefficients X_1, $X_2, \ldots, X_n$ are general.

In experimental terms we are accordingly not permitted to use such terms as beauty, good and truth with any absolute significance; comparative adjectives would be better, or "truer," and these only as applied to situations which did not differ widely or differed only in one or two elements. We may if we like take into account the method of transformation of one situation into another and compare two different "paths" which join the same two situations, as far as the equation (1) is applicable. Or we may, as many do, go outside experience altogether and introduce a hypothesis about the complete integrability of (1), trusting that experience will never yield such situations,—that is to say, such functions $X_1, X_2, \ldots X_n$,—as will contradict that hypothesis.

68. Further Generalization of the Utility Function.—Let us represent by $\dot{x}_1, \dot{x}_2, \ldots \dot{x}_n$ the respective rates of change of the quantities $x_1, x_2, \ldots, x_n$:

$$\dot{x}_1 = \frac{dx_1}{dt}, \dot{x}_2 = \frac{dx_2}{dt}, \ldots, \dot{x}_n = \frac{dx_n}{dt}.$$

The utility of a small change in the situation may involve also these quantities, since we may be concerned not only with changes in the quantities $x_1, x_2, \ldots, x_n$ themselves but also with their rates of consumption or acquisition. In fact it was with these rates of consumption or production that we were concerned in the earlier chapters. Is it immaterial to us, for instance, whether we take the entire Pasteur treatment in one dose or in several? For the equation (1) then we must often substitute the more general equation

$$
\begin{aligned}
X(x_1, \cdots \overset{x_n}{\cancel{\dot{y}_n}}, \dot{x}_1, \cdots \dot{x}_n)dx_1 + \cdots \\
+ X_n(x_1, \cdots x_n, \dot{x}_1, \cdots + \dot{x}_n)\, dx_n \\
+U_1(\dot{x}_1, \cdots, x_n, \dot{x}_1, \cdots x_n)d\dot{x}_1 + \cdots \\
+ U_n(x_1, \cdots x_n, \dot{x}_1, \cdots \dot{x}_n)dx_n = 0. \quad (6)
\end{aligned}
$$

This equation may be brought into the form (1), by introducing new variables $u_1, \cdots, u_n$, and written in the form

$$
\begin{aligned}
X_1(x_1, \cdots x_n, u_1, \cdots u_n)dx_1 + \cdots \\
+ X_n(x_1, \cdots x_n, u_1, \cdots, u_n)\, dx_n \\
+U_1(x_1, \cdots, x_n, u_1, \cdots u_n)du_1 + \cdots \\
+ U_n(x_1, \cdots, x_n, u_1, \cdots u_n)du_n = 0; \quad (7)
\end{aligned}
$$

but we must then introduce also the subsidiary relations

$$u_1 = \dot{x}_1,\ u_2 = \dot{x}_2 \cdots,\ u_n = \dot{x}_n. \qquad (7.1)$$

As we shall see, relations of even more general type than (7), (7.1) may be desirable as hypotheses.[1]

[1]The first economist to recognize the difficulties connected with the question of integrability was apparently Irving Fisher (see the citation given below on p. 166 in connection with Chap. XI).

CHAPTER XII

MARGINAL UTILITY

69. Exchange between Two Individuals.—The hypothesis of complete integrability or its equivalent in some form or other has been the basis of most of the work in theoretical economics, and a complete exposition of its consequences would accordingly occupy much more space than would be justified in a treatment of the brevity of ours. In this chapter will be given merely a fragmentary exposition, in order to suggest some of the directions in which the subject has been developed. Moreover something may be done directly from the equations (1) of the previous chapter without assuming that the infinitesimal loci of indifference can be assembled into a one parameter family of extended loci of indifference—this last being in fact exactly the hypothesis of complete integrability. For if there are such extended loci a utility function may be associated with this parameter. We consider first the case of two variables, where, as we have seen, no hypothesis of integrability is necessary.

We consider briefly the case of a single individual, and the two commodities x, y, subject to the equation:

$$X(x, y)dx + Y(x, y)dy = 0 \tag{1}$$

The simplest case is that where the variables are separable, that is, where

$$X(x, y) = X_1(x)Y_1(y), \; Y(x, y) = X_2(x)Y_2(y), \tag{1.1}$$

so that (1) may be written in the form

$$\frac{X_1(x)}{X_2(x)}dx + \frac{Y_2(y)}{Y_1(y)}dy = 0 \tag{1.2}$$

This equation may be directly integrated, and the solution written in the form

$$\int\frac{X_1(x)}{X_2(x)}dx + \int\frac{Y_2(y)}{Y_1(y)}dy = c.$$

Both of these integrals contain arbitrary constants, c_1 and c_2 respectively, and the value of the constant c that may be asso-

ciated with a given curve of indifference is arbitrary. Hence the value of the c may be adjusted in such a way that it increases as we pass from a curve of less desirability to one of greater, and as rapidly or as slowly as we please. With such choices of c we may write the solution of (1) in the form

$$U(x, y) = c \tag{2}$$

The curves (2) define the curves of indifference and the function $U(x, y)$ is a utility function for the individual.

Since it is only the ratios of the quantities X, Y in (1) which are important in determining the values of the variables x, y we may define any pair of functions of the form

$$\lambda(x, y)X(x, y), \ \lambda(x, y)Y(x, y)$$

as *marginal utilities* of the quantities x, y for the individual. Strictly speaking we should restrict the term to those pairs of functions which make the resulting differential expression

$$\lambda X dx + \lambda Y dy$$

a complete differential, that is, we should use the term only when functions $\lambda(x, y)$ are employed which are integrating factors of (1). Thus under the hypothesis (1.1) the coefficients in (1.2) may be described as marginal utilities in the strict sense, for (1.2) is a complete differential $dU(x, y)$. It is of the form

$$X(x)dx + Y(y)dy = 0,$$

and constitutes the special case where the marginal utility of x is a function of x alone and that of y is a function of y alone.

We turn now to a problem of exchange or barter between two individuals. We assume that x_1, y_1 are the variable and a_1, b_1 are the initial amounts of the two commodities in the possession of (1), and x_2, y_2; a_2, b_2 the corresponding quantities for (2), and that the respective marginal utility equations are the following

$$\begin{aligned} X_1(x_1, y_1)dx_1 + Y_1(x_1, y_1)dy_1 &= 0 \\ X_2(x_2, y_2)dx_2 + Y_2(x_2, y_2)dy_2 &= 0, \end{aligned} \tag{3}$$

with the respective utilities increasing as we move across these curves away from the origin. If we think of the utility function as the height $U(x, y)$, over a point (x, y) of the base, of a hill whose contour lines are the curves of indifference, the object of each individual in making the exchange is to climb as far up on his own utility hill as possible. The final locations are not

independent; in fact, the whole process of exchange is subject to the conditions that the total amounts of the two quantities are unchanged

$$\begin{aligned} x_1 + x_2 &= a_1 + a_2 \\ y_1 + y_2 &= b_1 + b_2, \end{aligned} \tag{4}$$

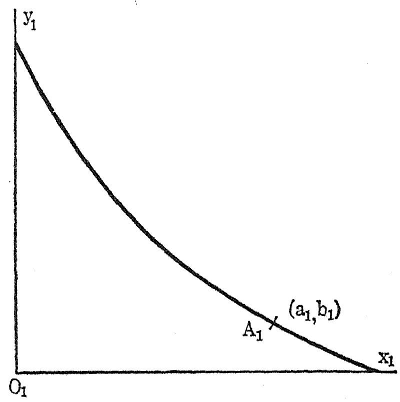

FIG. 24.

and that none of them become negative. The process of exchange can be profitably continued, bit by bit, if we like, as long as

$$\begin{aligned} X_1 dx_1 + Y_1 dy_1 &\geqq 0 \\ X_2 dx_2 + Y_2 dy_2 &\geqq 0. \end{aligned} \tag{4.1}$$

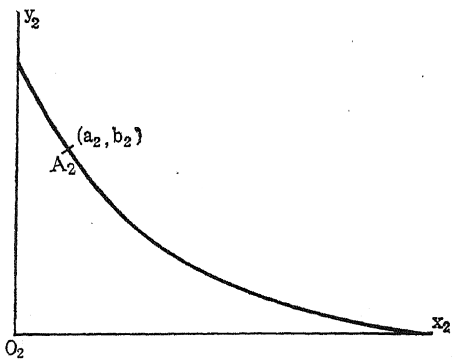

FIG. 25.

Let us suppose that the indifference curve for (1) through the point (a_1, b_1) is of the form illustrated in Fig. 24 with a similar curve for (2) as in Fig. 25.

These curves may be profitably superimposed on a single diagram, as in Fig. 26, in order to maintain the relations (4). Here the coordinates of O_2 with respect to O_1 are $(a_1 + a_2, b_1 + b_2)$; and these are also the coordinates of O_1 with respect to O_2; and the points A_1 and A_2 are superimposed at A.

Any curve leading away from A will be a possible path of exchange from the given situation, but only those which lead into the vertically shaded region will be acceptable to both individuals, since only those will yield a final utility for each

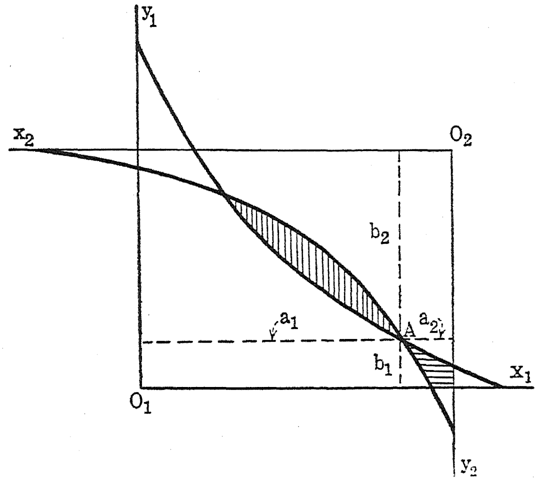

FIG. 26.

one which is greater than the initial utility. We may assume that a final position will be acceptable only if it yields an actual increase of utility for both individuals, that is to say, if the final point lies actually within this shaded region. But what final point will be most likely? In order to decide on this point an additional hypothesis is necessary.

From (4) we have $dx_1 = -dx_2$, $dy_1 = -dy_2$, and so we may write

$$\frac{dy_1}{dx_1} = \frac{dy_2}{dx_2} = -p, \tag{5}$$

where p is accordingly the price of unit of x in terms of y, or, what amounts to the same thing, the ratio of the prices of units of x and y respectively in terms of money. If the price remains constant during the exchange the relation (5) becomes

$$\frac{y_1 - b_1}{x_1 - a_1} = \frac{y_2 - b_2}{x_2 - a_2} = -p, \tag{5.1}$$

Here p is also to be determined. The equations

$$\frac{y_1 - b_1}{x_1 - a_1} = -p, \frac{y_2 - b_2}{x_2 - a_2} = -p,$$

evidently represent the same straight line in Fig. 26, p constant, referred in the one case to the x_1, y_1 axes and in the other to the x_2, y_2 axes, for the point (x_1, y_1) is the same as (x_2, y_2). The straight line passes through A.

If the exchange is carried out at constant price, it will proceed until the price line just mentioned becomes tangent to some indifference curve of (1) or (2), these curves being imagined as roughly parallel to those given in Fig. 26. Otherwise carrying the transaction further will increase the desirability for both (1) and (2). If the point reached is A_1, a new exchange will be arranged with a new price line through A_1, for such a line may still be drawn so as to increase the utilities for (1) and (2). Thus the bargain will be continued through successive points A_1, A_2, . . . until a point P is reached where the last price line drawn becomes tangent to indifference curves for (1) and (2) at the same time.

We assume that the transaction is carried out at one stage and at such a price p that it will be completed at a point P where both curves of indifference are tangent to the price line. The point of equilibrium will then be one where a curve of indifference for (1) is tangent to a curve of indifference for (2) and the common tangent passes through A.

If the dx_2, dy_2 are respectively replaced by $-dx_1$, $-dy_1$ in (4.1), and the signs $\geq$ replaced by $=$ signs on account of the fact that the path of transformation is tangent to the indifference curves at the point in question, these equations become

$$X_1(x_1, y_1)dx_1 + Y_1(x_1, y_1)dy_1 = 0$$

$$X_2(a_1 + a_2 - x_1, b_1 + b_2 - y_1)dx_1 + Y_2(a_1 + a_2 - x_1, b_1 + b_2 - y_1)dy_1 = 0$$

whence

$$\frac{X_1(x_1\ y_1)}{X_2(a_1 + a_2 - x_1, b_1 + b_2 - y_1)} = \frac{Y_1(x_1, y_1)}{Y_2(a_1 + a_2 - x_1, b_1 + b_2 - y_1)} \tag{6}$$

Also from (5) and (5.1) we have

$$\frac{y_1 - b_1}{x_1 - a_1} = \frac{-X_1(x_1, y_1)}{Y_1(x_1, y_1)} \tag{6.1}$$

since the slope p of the price line is the same as the slope dy_1/dx_1 of the tangent to the indifference curve for (1) at (x_1, y_1).

These furnish two equations, in general, to determine x_1, y_1; and when these are determined the equations (4) and (5.1) may be used to calculate the quantities x_2, y_2 and p. But this point is reached only on the hypothesis that the trade is carried out at one price; and if this part of the assumption is dropped, any point P is the vertically shaded region may illustrate the final position, provided that it is a point where a curve of indifference of (1) is tangent to a curve of indifference of (2).

Other than these special hypotheses which have been made explicitly there is another which we have made, tacitly, by drawing the specimen curves of indifference in Figs. 24 and 25 as convex towards the origin. If the curves happen to be concave towards the origin we may still solve exactly the same equations as before, but they no longer have any meaning with reference to the problem. This second situation may be represented geometrically by imagining that the curves belonging to (1) and (2) in Fig. 26 are interchanged. The paths which lead from A and now represent favorable transactions will be only those which lead downward to the right, into the horizontally shaded region; and *the transaction may be carried on with an increasing utility to both individuals until one of the quantities x_1, y_1, x_2 or y_2 becomes zero.* Such indifference curves as these just mentioned are typical perhaps for an individual who seeks to corner a market, or for a collector.

70. Exchange between Two Individuals for Three Commodities.—Let us extend our analysis now to three commodities, the individual (1) possessing amounts x_1, y_1, z_1 respectively of them, and the individual (2) amounts x_2, y_2, z_2. The initial values of these quantities for the two individuals may be denoted by a_1, b_1, c_1 and a_2, b_2, c_2 respectively. The utility conditions analogous to (3) become

$$\begin{aligned} X_1(x_1, y_1, z_1)dx_1 + Y_1(x_1, y_1, z_1)dy_1 + Z_1(x_1, y_1, z_1)dz_1 \geq 0 \\ X_2(x_2, y_2, z_2)dx_2 + Y_2(x_2, y_2, z_2)dy_2 + Z_2(x_2, y_2, z_2)dz \geq 0 \end{aligned} \qquad (7)$$

If the amount of z exchanged is regarded as a function of the amounts x and y, for each individual, and p and q denote respectively the prices of x and y in terms of z, we have

$$-p = \frac{\partial z_1}{\partial x_1} = \frac{\partial z_2}{\partial x_2}; \ -q = \frac{\partial z_1}{\partial y_1} = \frac{\partial z_2}{\partial y_2}; \qquad (7.1)$$

in fact $-p$ is the change Δz_1 of z_1 when the change Δx_1 of x_1 is 1 and y_1 is held constant, if we assume that p is constant, during the change Δx_1.

Also, the total amounts of x, y and z remain constant

$$x_1 + x_2 = a_1 + a_2,\ y_1 + y_2 = b_1 + b_2,\ z_1 + z_2 = c_1 + c_2. \tag{7.2}$$

If, as before, an exchange takes place at constant price from the initial position it will be governed by the equations

$$\begin{aligned} z_1 - c_1 &= p(x_1 - a_1) + q(y_1 - b_1) \\ z_2 - c_2 &= p(x_2 - a_2) + q(y_2 - b_2), \end{aligned} \tag{8}$$

of which the second is a consequence of the first, on account of (7.2). And if we say now that the planar elements obtained by using the equality signs in (7) shall lie in the respective planes of transaction (8), we have, using also (7.2),

$$\begin{aligned} p &= -\frac{X_1(x_1, y_1, z_1)}{Z_1(x_1, y_1, z_1)} = \\ &\qquad -\frac{X_2(a_1 + a_2 - x_1, b_1 + b_2 - y_1, c_1 + c_2 - z_1)}{Z_2(a_1 + a_2 - x_1, b_1 + b_2 - y_1, c_1 + c_2 - z_1)} \\ q &= -\frac{Y_1(x_1, y_1, z_1)}{Z_1(x_1, y_1, z_1)} = \\ &\qquad -\frac{Y_2(a_1 + a_2 - x_1, b_1 + b_2 - y_1, c_1 + c_2 - z_1)}{Z_2(a_1 + a_2 - x_1, b_1 + b_2 - y_1, c_1 + c_2 - z_1)} \end{aligned} \tag{9}$$

The equations (9) and the first of equations (8) furnish five equations to determine the five quantities x_1, y_1, z_1, p, q. And if these equations happen not to be dependent or contradictory the problem may be regarded as solvable, and the remaining unknowns may be determined by (7.2).

No hypothesis of integrability with regard to the equations (7) has been introduced, or is necessary in order to get a mathematical solution. But so far, it is merely a mathematical problem which has been solved. For from all that is said it is not evident that the transaction involves any increase of utility for either individual. On the other hand, if we assume that the planar elements, obtained by using the equal signs in (7), envelope surfaces of indifference (that is, that those differential expressions are completely integrable), which are convex towards the origin, something of the same analysis may be carried out that was described in the case of two commodities. In fact, a favorable transaction, with constant prices p and q, will have for geometrical image a curve traced in the plane, given by (8), provided

this path leads into the region between the two surfaces which corresponds to increased utility. And if the transaction yields an equilibrium point beyond which no further exchange will take place, the coordinates of this point and the prices will satisfy (7.2), (8) and (9).

If we do not make the hypothesis of complete integrability, we must suppose that the path of the transaction leads from the initial point A in such a way as to correspond to an increase of utility; in other words, the quantities dx_1, dy_1, dz_1, dx_2, dy_2, dz_2 at every point of the path of transaction must satisfy (7). The path must always cross the infinitesimal planes of indifference in the proper sense. If then it is possible to make such a path by a straight line in such a way as to reach a point of equilibrium beyond which no further transaction will take place, the coordinates of this point and the prices p and q will again satisfy the equations (7.2), (8) and (9). But we cannot say that the gain of "utility" will be the same as if we reached this same equilibrium point by some other path of transaction—still less, that every individual can maximize his satisfaction at the same time!

There are special cases in which we know that the equations of the infinitesimal loci of indfference are completely integrable. Such is the case where the marginal utility of each commodity depends on the amount of that commodity alone. The equations become then

$$X_1(x_1)dx_1 + Y_1(y_1)dy_1 + Z_1(z_1)dz_1 = 0$$
$$X_2(x_2)dx_2 + Y_2(y_2)dy_2 + Z_2(z_2)dz_2 = 0,$$

and the surfaces of indifference are given by the equations

$$\int X_1(x_1)dx_1 + \int Y_1(y_1)dy_1 + \int Z_1(z_1)dz_1 = c_1$$
$$\int X_2(x_2)dx_2 + \int Y_2(y_2)dy_2 + \int Z_2(z_2)dz_2 = c_2.$$

But, going to the other extreme, we may also meet the case where each marginal utility is a function of the six variables x_1, x_2, y_1, y_2, z_1, z_2, and possibly also of the time, of the prices and of other quantities $\dot{x}_1$, . . . , etc.

71. General Exercises.

1. Given the marginal utility equations

$$x_1dy_1 + y_1dx_1 \geq 0,\ x_2dy_2 + y_2dx_2 \geq 0,$$

for which the indifference curves are the rectangular hyperbolas

$$x_1y_1 = c_1 \text{ and } x_2y_2 = c_2,$$

and given initial amounts a_1, b_1 and a_2, b_2 for the individuals (1) and (2) respectively, show that the equilibrium point for a single price transaction is given by

$$x_1 = \frac{a_1 + pb_1}{2p}, \quad y_1 = \frac{a_1 + pb_1}{2}$$

where

$$p = \left(\frac{b_1 + b_2}{a_1 + a_2}\right)$$

2. Discuss the following situations by drawing diagrams corresponding to Fig. 26 and indicating the possible paths of favorable transaction:

$$dx_1 + dy_1 \geq 0, \quad dx_2 + dy_2 \geq 0 \tag{a}$$
$$dx_1 + 2dy_1 \geq 0, \quad 2dx_2 + dy_2 \geq 0 \tag{b}$$
$$2dx_1 + dy_1 \geq 0, \quad dx_2 + 2dy_2 \geq 0 \tag{c}$$

3. In equation (3) of the text the functions X_1, Y_1, X_2, Y_2 are supposed to be single valued functions of x_1, y_1, etc. Can two indifference curves for the individual (1), corresponding to two different values of the utility function $U(x_1, y_1)$, intersect at an angle? Can they touch each other?

Discuss the indifference curves for

$$(x_1 - 3)dy_1 - (y_1 - 2)dx_1 \geq 0$$

by solving the differential equation

$$\frac{dy_1}{dx_1} = \frac{y_1 - 2}{x_1 - 3}$$

or by plotting it as in Chapter IV. Would this situation correspond to a practical case?

4. Are there indifference surfaces in the case where the infinitesimal loci of indifference (planar elements) are given by

$$xydx + y^2dy + z^2dz = 0?$$

Are there such surfaces for

$$xdx + ydy + zdz = 0? \text{ for } ydx - dz = 0?$$

CHAPTER XIII

THE THEORY OF PRODUCTION

72. Exchange of Rates of Production.—The theory of exchange, based explicitly or tacitly on the hypothesis of complete integrability, as we have outlined it in the previous chapter, has caused a separation of the theory of exchange from the theory of production, quite different from the theory expounded in the early chapters. This separation is in line with the classical theories of economics which are not given in mathematical form, and it may be regarded as an advantage. However, both kinds of theories may in a measure be combined into one, if for the quantities x_i, y_i, . . . of Secs. **69** and **70** are substituted their rates of acquisition u_i, v_i, . . . , by production or exchange, for each individual.

In fact, if we suppose that constant total amounts a, b are produced in unit time, u_1, v_1 and u_2, v_2 being the number of units of the commodities which come into the possession of two individuals (1) and (2) respectively in unit time, we shall have

$$u_1 + u_2 = a_1 + a_2 = a, \; v_1 + v_2 = b_1 + b_2 = b \tag{1}$$

where a_1, a_2 and b_1, b_2 are the values of the quantities a, b at $t = t_0$. We may suppose that the producers will exchange portions of their productions so as finally to have amounts which are more suitable for consumption.

If we denote δu_1, δv_1, . . . arbitrary small changes in the u_i, v_i, . . . which may be made at a given instant, the possible changes of that kind which are advantageous may be assumed to be governed by the inequalities

$$\begin{aligned} X_1(u_1, v_1)\delta u_1 + Y_1(u_1, v_1)\delta v_1 &\geq 0 \\ X_2(u_2, v_2)\delta u_2 + Y_2(u_2, v_2)\delta v_2 &\geq 0. \end{aligned} \tag{1.1}$$

The price for such possible transactions would be given by

$$-\frac{\delta v_1}{\delta u_1} = -\frac{\delta v_2}{\delta u_2} = p, \tag{1.2}$$

since $|\delta v_1|$ units of y per unit time would be exchanged against $|\delta u_1|$ units of x per unit time.

For the infinitesimal changes which correspond to the formation of the actual transaction we may use the symbols du_1, dv_1, . . . And if the transaction occurs at constant price we shall have

$$\frac{v_1 - b_1}{u_1 - a_1} = \frac{v_2 - b_2}{u_2 - a_2} = -p$$

since

$$\frac{dv_1}{du_1} = \frac{dv_2}{du_2} = -p$$

From here on the analysis is the same as that given in the previous chapter and the coordinates of the point of equilibrium for the transaction at constant price will be determined by the equations

$$\begin{aligned} X_1(u_1, v_1)du_1 + Y_1(u_1, v_1)dv_1 &= 0 \\ X_2(u_2, v_2)du_2 + Y_2(u_2, v_2)dv_2 &= 0, \end{aligned}$$

in conjunction with those already given. By elimination then we obtain the equations

$$\frac{X_2(a - u_1, b - v_1)}{Y_2(a - u_1, b - v_1)} = \frac{X_1(u_1, v_1)}{Y_1(u_1, v_1)} = -\frac{v_1 - b_1}{u_1 - a_1}, \tag{2}$$

which are two relations in (u_1, v_1) similar to (6.1), Chapter XII. The value of each member of (2) is p, and the u_2 and v_2 are then given by (1).

In this analysis it is immaterial whether the entire transaction is carried out at one instant or spread over an interval of time of arbitrary length, for the rates of production of the commodities for each individual are given as constants. But there is no essential change if the amounts manufactured are variable, but known as functions of the time t, and the transaction is to be completed at a given time t_1.

In this case we let the total productions be $a(t)$, $b(t)$, per unit time, and denote by $a_i(t)$, $b_i(t)$ the quantities which would measure the productions per unit time of x, y for the individual (i) if there were no exchange. Using the same signification for the quantities δu_i, δv_i the inequalities (1.1) remain as before, but for the actual changes that take place, the equations are

$$-\frac{d(v_1 - b_1)}{d(u_1 - a_1)} = -\frac{d(v_2 - b_2)}{d(u_2 - a_2)} = p$$

which for a transaction at constant price yields

$$v_1(t) - b_1(t) = -p(u_1(t) - a_1(t)) + c_1$$

with a similar relation for the individual (2). If it is assumed that initially there is no exchange, that is, that

$$u_i(t_0) = a_i(t_0),\ v_i(t_0) = b_i(t_0),\ i = 1, 2,$$

the equations of transaction reduce to their earlier form

$$\frac{v_i(t) - b_i(t)}{u_i(t) - a_i(t)} = -p = \text{constant},\ i = 1, 2. \tag{2.1}$$

For the position of equilibrium we have these equations and the equations which say that the line of transaction is tangent to the curves of indifference, namely

$$X_i d(u_i - a_i) + Y_i d(v_i - b_i) = 0,\ i = 1, 2, \tag{3}$$

for the arbitrary small instantaneous exchanges δu_i, δv_i in (1.1) are replaced now by the quantities $d(u_i - a_i)$, $d(v_i - b_i)$, which are the actual exchanges in the time dt. But the combination of (3) with the earlier equations yield again exactly the equations (2), in which the various quantities take their respective values at the time t_1.

The analysis may be still further generalized without altering the final solution if we assume that the X_i, Y_i are functions of the a_i, b_i, as well as of the u_i, v_i, and even of t; but here the significance of the problem is apt to be lost without some assumption of integrability.

EXERCISE.—Assuming that initially we have

$$u_i(t_0) = a_i(t_0) + \alpha_i,\ v_i(t_0) = b_i(t_0) + \beta_i$$

where the α_i, β_i are given constants, and that the transaction is made at constant price, deduce the equation of transaction corresponding to (2.1).

What are the quantities which must not become negative?

73. Factors of Production.—Suppose that in order to manufacture a quantity a of a certain commodity per unit time, a manufacturer needs to employ quantities $\alpha, \beta, \gamma \ldots$ of certain raw materials, other manufactured articles, services of machines, of money, of labor, etc. The quantities $\alpha, \beta, \gamma \ldots$ may be called factors of production. The simplest assumption is that these quantities depend merely on the amount a which is produced, so that they are functions of the one variable a.

$$\alpha = \alpha(a) \quad \beta = \beta(a) \quad \gamma = \gamma(a), \cdots \tag{4}$$

The cost q of producing a per unit time will then be the sum of the

costs of the α, β, γ . . . , and therefore will be a function of a which may be written in the form

$$q(a) = q_\alpha + q_\beta + \cdot \cdot \cdot = p_\alpha \alpha + p_\beta \beta + \cdot \cdot \cdot \tag{4.1}$$

where p_α, p_β etc., are the respective prices or costs to the manufacturer of the amounts α, β, . . .

But often there are several different ways of combining the amounts of primary elements α, β, . . . in order to produce the manufactured element a, and instead of the simple relations (4) we may have one or more equations of the form

$$\begin{aligned} f(a, \alpha, \beta, \cdot \cdot \cdot) &= 0 \\ g(a, \alpha, \beta, \cdot \cdot \cdot) &= 0 \\ . \quad . \quad . \quad . \quad &. \quad . \quad . \\ . \quad . \quad . \quad . \quad &. \quad . \quad . \end{aligned} \tag{5}$$

which express the relations involved between the quantities required in the technical process of manufacture. On this hypothesis, if there are k factors of production there will have to be less than k equations (5); for if there were k such independent equations we could regard them as solved for the quantities α, β, . . . in terms of a, and we should thus revert to the original case. The equations (4) would then again appear.

In order to analyze the new situation let us assume that there are two primary elements α, β, and one relation of the form (5), namely

$$f(a, \alpha, \beta) = 0, \tag{6}$$

which for small changes of the variables takes also the form

$$\frac{\partial f}{\partial a} da + \frac{\partial f}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \beta} d\beta = 0. \tag{6.1}$$

We may suppose that a definite amount a is to be manufactured. Then $da = 0$ and (6.1) becomes

$$\frac{\partial f}{\partial \alpha} d\alpha + \frac{\partial f}{\partial \beta} d\beta = 0. \tag{6.2}$$

The cost q of a will no longer be fixed as in (4.1), but will depend on the relative amounts of α, β that are used:

$$q = q(\alpha, \beta) = q_\alpha + q_\beta \tag{7}$$

If we make the natural assumption that this cost is to be a minimum, we shall have $dq = 0$, or

$$\frac{\partial q}{\partial \alpha} d\alpha + \frac{\partial q}{\partial \beta} d\beta = 0 \tag{7.1}$$

The equations (6.2) and (7) together yield the following relation:

$$\frac{\dfrac{\partial f}{\partial \alpha}}{\dfrac{\partial f}{\partial \beta}} = \frac{\dfrac{\partial q}{\partial \alpha}}{\dfrac{\partial q}{\partial \beta}} \tag{8}$$

The equations (6), with a fixed, and (7) are assumed to be independent, since otherwise q would be already determined by giving a; hence (6.2) and (7.1) are independent, and (6) and (8) furnish two independent equations to determine α and β.

The equation (8) has a simple interpretation. If the prices p_α and p_β to the manufacturer are independent of the amounts which are bought by him in unit time, the quantities $\partial q/\partial \alpha$ and $\partial q/\partial \beta$ will be constant and equal to those prices respectively:

$$\frac{\partial q}{\partial \alpha} = p_\alpha, \quad \frac{\partial q}{\partial \beta} = p_\beta$$

For we have

$$q_\alpha = \alpha p_\alpha, \; q_\beta = \beta p_\beta$$

with p_α and p_β constants. On the other hand, both of the members of (8) are equal to the quantity

$$-\frac{d\beta}{d\alpha} = \frac{|d\beta|}{|d\alpha|},$$

where, according to (6.2), $d\alpha$ and $d\beta$ are the small changes of α and β such that a substitution of $d\alpha$ for $d\beta$ will not affect the amount a produced. If we name these quantities *corresponding marginal factors of production*, we may say that *optimum arrangement is characterized by having the numerical values of the corresponding marginal factors of production proportional to the constant prices of the respective factors of production.*

If the prices p_α and p_β depend on the total quantities α and β that are brought by the manufacturer, we have

$$p_\alpha = \frac{q_\alpha}{\alpha}, \; p_\beta = \frac{q_\beta}{\beta}$$

and the quantities $\partial q/\partial \alpha$, $\partial q/\partial \beta$ are the marginal unit costs to the manufacturer of α and β, in the sense of Chapter I. Since $q_\alpha = \alpha p_\alpha$ we have

$$\frac{\partial q_\alpha}{\partial \alpha} = p_\alpha + \alpha \frac{\partial p_\alpha}{\partial \alpha}$$

In this case the optimum situation is characterized by having the numerical values of the corresponding marginal factors of production

for α and β proportional respectively to the marginal unit costs of α and β to the manufacturer.

The equation (8) may be interpreted in another way. If we solve (6) for a

$$a = F(\alpha, \beta)$$

we have

$$da = \frac{\partial F}{\partial \alpha}d\alpha + \frac{\partial F}{\partial \beta}d\beta$$

We denote the quantities $\partial F/\partial \alpha$, $\partial F/\partial \beta$, which are sometimes called coefficients of production, by F_α' and F_β' respectively, and the quantities $\partial q/\partial \alpha$, $\partial q/\partial \beta$ by q_α', q_β' respectively. Then if β is held constant $da = \delta a_\alpha = F_\alpha' d\alpha$, where δa_α is the increment in a due to a small change in α alone. The equation (8) then takes the form,

$$\frac{F_\alpha'}{q_\alpha'} = \frac{F_\beta'}{q_\beta'} \quad \text{or} \quad \frac{F_\alpha' d\alpha}{q_\alpha' d\alpha} = \frac{F_\beta' d\beta}{q_\beta' d\beta},$$

or

$$\frac{\delta a_\alpha}{\delta q_\alpha} = \frac{\delta a_\beta}{\delta q_\beta}$$

In the optimum situation the increments of "a" due to changing each of the factors of production separately are proportional to the increments of cost of these factors.

In this form the theorem just proved is valid for more than two factors of production provided that there is still only one equation (5). If we write this in the form

$$a = F(\alpha, \beta, \gamma)$$

the equation corresponding to (6.2) is

$$F_\alpha' d\alpha + F_\beta' d\beta + F_\gamma' d\gamma = 0$$

where $d\alpha$ and $d\beta$, say, may be regarded as independent and $d\gamma$ as determined by this relation.

For the optimum situation

$$q_\alpha' d\alpha + q_\beta' d\beta + q_\gamma' d\gamma = 0$$

If we eliminate $d\gamma$, from these two equations, we have

$$\frac{F_\alpha'}{F_\gamma'}d\alpha + \frac{F_\beta'}{F_\gamma'}d\beta = \frac{q_\alpha'}{q_\gamma'}d\alpha + \frac{q_\beta'}{q_\gamma'}d\beta,$$

and since the $d\alpha$ and $d\beta$ are arbitrary, we have

$$\frac{F_\alpha'}{F_\gamma'} = \frac{q_\alpha'}{q_\gamma'}, \; \frac{F_\beta'}{F_\gamma'} = \frac{q_\beta'}{q_\gamma'}$$

Hence

$$\frac{F_\alpha'}{q_\alpha'} = \frac{F_\beta'}{q_\beta'} = \frac{F_\gamma'}{q_\gamma'}$$

or

$$\delta a_\alpha : \delta a_\beta : \delta a_\gamma = \delta q_\alpha : \delta q_\beta : \delta q_\gamma$$

EXERCISE.—Prove the corresponding relation for k factors of production, with one equation (5).

It will occur to the reader that not only are the factors of production connected by relations of the type (5), but also that several different commodities may result from the same process of manufacture. For the equations (5) then we may substitute the more general equations

$$\begin{aligned} f(a, b, c; \alpha, \beta, \cdots) &= 0 \\ g(a, b, c; \alpha, \beta, \cdots) &= 0 \\ \cdots\cdots&\cdots \\ \cdots\cdots&\cdots \end{aligned} \tag{9}$$

The cost of manufacture will then be a function $q(\alpha, \beta, \ldots)$ which is to be rendered a minimum. The problem may in part be avoided by considering the extra quantities $b, c, \ldots,$ as factors of production in a; for a given value of a the quantity to be made a minimum would be

$$q(\alpha, \beta, \cdots) - p_b b(t) - p_c c(t) - \cdots,$$

in which $p_b b(t)$ is the selling value of b, etc., provided that the prices p_b, p_c etc., are given. The $b, c, \ldots$ are here regarded as waste products in the manufacture of a.

74. Determination of the Rates of Production.—Given the rates of production $a_i(t)$, $b_i(t)$, . . . of various products of manufacture by various individuals, or corporations or groups of corporations, we have been able on the one hand to construct a theory of how these rates of productions are exchanged, in terms of a utility function or marginal utilities, and on the other hand to discuss the allocation of raw materials or factors of production, with reference to the manufacture, in terms of minimum cost. Here again our ideas may be slightly generalized by substituting for the cost some other utility function, since it need not always be the cost of manufacture of a given amount which the manufacturer desires to make as small as possible. Moreover since the raw materials of one product are the products of some other industry, it is apparent that we have been able to outline in

Secs. **72** and **73** a rather general theory. But what is it that can be used to determine the actual rates of production? If for instance all the given relations should happen to be linear and homogenous in the variables, they would still be satisfied when all the values of the variables are doubled, or multiplied by an arbitrary constant, and if the relations were not linear there would nevertheless be other values of the variables, containing arbitrary quantities, which would continue to satisfy them.

The answer to the question is given in terms of the sort of analysis that we met in Chapters I and III. The various producers, according to some assumed sort of business organization—monopoly, competition or what-not—seek to make their profits a maximum, or to make some other quantity a maximum with relation to their roles in the community as consumers, misers, dynasts or philanthropists.

It may seem not a little miraculous that in the mathematical treatments of economics, as suggested in these last few chapters, there should always be just the right number of equations to determine all the unknowns; but the secret is precisely here. Suppose for instance, that, as a result of the kind of analysis given in Secs. **72, 73**, there are left m variables still undetermined. We may then introduce another utility function, containing m or more variables, and by making it a maximum or minimum obtain just the m equations desired. For by eliminating the quantities that are known in terms of the others, the function is reduced to one of the m variables alone, say $U(x_1, x_2, \ldots, x_m)$, and by making it a maximum or minimum we obtain the m equations

$$\frac{\partial U}{\partial x_1} = 0, \frac{\partial U}{\partial x_2} = 0, \cdots, \frac{\partial U}{\partial x_m} = 0.$$

Or we may introduce several such functions $U_1, \ldots, U_h$, and assume that U_1 is to be made a maximum when only the variables of a subset of the $x_1, \ldots, x_m$ are allowed to change, U_2 a maximum when those of second subset are allowed to vary, and so on. In a rough way, we have here the distinction between the monopoly or cooperation hypotheses and the competition hypotheses, of Chapters I and III. The choice and investigation of these functions is the methodology of theoretical economics, from this point of view.

It is nevertheless questionable if our analysis is not too much simplified to take account of the problems which we most desire

to cover. In order to discuss this point let us consider briefly the relation of the problem of Chapter I to the present analysis. We may limit ourselves to two individuals, (1), a producer, and (2), a consumer, and discuss the exchange between them with its resulting equilibrium. For simplicity we may suppose that the quantities v, b refer to money, u, a to the commodity in question, and suppose that (1) exchanges goods for money, and (2) money for goods. We may take the X_i, Y_i as functions of the corresponding a_i, b_i, u_i, v_i.

If the individual (2) produces nothing of the commodity, but is ready to buy u_2, the price being taken as p and the income b_2 being known, we may write

$$a_2 = 0,\ v_2 - b_2 = -pu_2,\ v_2 = b_2 - pu_2.$$

The equation which is to be satisfied if the utility for (2) is a maximum is

$$X_2(b_2, u_2, v_2)d(u_2 - a_2) + Y_1(b_2, u_2, v_2)d(v_2 - b_2) = 0$$

or

$$X_2(b_2, u_2, b_2 - pu_2) - Y_2(b_2, u_2, b_2 - pu_2)p = 0. \qquad (10)$$

When we regard this as solved for u_2, we have an equation of the form

$$u_2 = f(p), \qquad (10.1)$$

which expresses u_2 in terms of a typical demand function.

Similarly, if (1) assumes that he will make only what he will sell, and we consider only the money received by selling to (2), we shall have

$$u_1 = 0,\ b_1 = 0,\ v_1 = pa_1,$$

and the equation for maximum utility for (1) becomes

$$X_1(a_1, pa_1) - Y_1(a_1, pa_1)p = 0 \qquad (11)$$

Solved for a, this becomes also an equation of the form

$$a_1 = g(p) \qquad (11.1)$$

The position of equilibrium is obtained by writing

$$a_1 = u_2$$

whence

$$f(p) = g(p)$$

and we have an equation to determine p.

But the equation (11.1) is in the form of an offer function, which occurred in Chapter I in connection with the problem of

price fixing, but not in the general problem of monopoly. In the latter case, and with reference to the present notation, we had

$$\frac{d\pi}{da_1} = 0 = \frac{d}{da_1}(pa_1 - q(a_1))$$

or

$$p + a_1\frac{dp}{da_1} - q'(a_1) = 0.$$

Hence a_1 cannot be expressed in terms of p without knowing dp/da_1, a quantity which can be calculated only with knowledge of the demand function. We should have

$$da_1 = du_2 = f'(p)dp, \frac{dp}{da_1} = \frac{1}{f'(p)}.$$

Accordingly, if we indicate by $[f]$ the coefficients which may be used in the construction of the function $f(p)$, the equation of indifference for (1) will be of the following type:

$$X_1(a_1, pa_1, [f]) - Y_1(a_1, pa_1, [f])p = 0 \tag{11.2}$$

And this is of quite different character from those considered in Secs. **71, 72.**

If, as in Chapter IV, the demand depends on the rate of change of price dp/dt as well as on the price, the marginal utility equation for (2) will be of form

$$X_2\delta u_2 + Y_2\delta v_2 + P_2\delta p + T_2\delta t = 0,$$

which is again of novel character with regard to the present analysis. However in connection with this type of law of demand there are problems of greater interest than any we have heretofore considered, and we turn now to these and to simpler methods of analysis.

75. General Exercises.

1. If equations (5) have the form

$$a = F(\alpha, \beta)$$
$$b = G(\alpha, \beta)$$

and the cost is given by $q = q_\alpha(\alpha) + q_\beta(\beta)$, discuss the case where a is given and the price of b is given as a constant p_b. Derive the equation

$$\frac{q_\alpha' - p_bG_\alpha'}{F_\alpha'} = \frac{q_\beta' - p_bG_\beta'}{F_\beta'}$$

Does this give us enough information to determine the solution?

2. Discuss the case where a and b are both given.

3. Suppose that there are two equations (5) of the form

$$f(a, \alpha, \beta, \gamma) = 0$$
$$g(a, \alpha, \beta, \gamma) = 0$$

and that the cost is given as a function $q(\alpha, \beta, \gamma)$. Find equations to determine the optimum situation, when a is given.

CHAPTER XIV

A PROBLEM IN ECONOMIC DYNAMICS

76. Total Profit during an Interval of Time.—Heretofore, except for a small part of Chapter IV, we have been concerned with problems of equilibrium; and to about the same extent this has been the limit of the classical theories of economics, mathematical or otherwise. It is essential now however to abandon that point of view and consider possible processes by means of which one situation may be changed into another. Such processes, although the term is not altogether happy on account of the specific reference to a subject of mechanics and on account of the artificiality of method which a loose analogy suggests, may be called processes of *economic dynamics*.

We consider again, as in Chapter I, a single producer for whom the cost of producing u units of a given commodity per unit time is a quadratic function of the amount u produced:

$$q(u) = Au^2 + Bu + C, \tag{1}$$

with the coefficients A, B, C all positive

$$A > 0, B > 0, C > 0. \tag{1.1}$$

For the demand function we take, as in Chapter IV, a linear differential expression

$$y = ap + b + h\frac{dp}{dt}, \tag{2}$$

the term involving dp/dt being introduced in order to take account of the fact that the demand for a commodity may depend on the rate of change of the price (whether, for instance, it is increasing or decreasing) as well as on the price itself. Here we shall take, as before,

$$a < 0, b > 0$$

and leave, for the present, the sign of h arbitrary, although the large number of "lambs" in existence would indicate that the practical case would be to take h positive. The quantities p, u, y will be considered as variable in time, and we wish to

determine them as functions of t. We shall assume that the producer regulates u so that $u = y$.

The rate of profit at any time (that is, the profit per unit time if the profit is uniform) is given by the formula

$$\pi = py - q(u) = pu - q(u) \tag{3}$$

and the total profit over an interval of time (t_0, t_1), from $t = t_0$ to $t = t_1$, will be given accordingly by the expression

$$\Pi = \int_{t_0}^{t_1} \pi(p, p')dt = \int_{t_0}^{t_1} \{p(ap + b + hp') \\ - A(ap + b + hp')^2 - B(ap + b + hp') - C\}dt, \tag{4}$$

in which p' denotes dp/dt. A simple assumption, descriptive of monopoly, is *to make a maximum the total profit over an interval of time*. We can imagine that at the initial time t_0 the cost function has been changed from some previous formula to that given above, and the producer desires, on the basis of the new cost function, to arrive at a new equilibrium price, already calculated, at the time t_1, by continuous change and in the most profitable manner. We can then assume as a first problem that *the initial and final values of p are given as p_0 and p_1 respectively*, and that the problem is to determine $p(t)$ in the intervening time so as to make π a maximum. Since it turns out that we shall need the second derivatives of $p(t)$ we may as well assume that $p(t)$ is continuous, with its first and second derivatives, $p'(t)$ and $p''(t)$.

77. Maximum of an Integral.—The problem of finding a function $p(t)$ which will maximize (4) can be solved by reducing it to the principal problem of elementary differential calculus, that is, to maximize a function of a single variable x. We introduce the variable x as follows.

Let $f(t)$ be the function which we wish to determine as the optimum choice of $p(t)$, assuming for the moment that there is one which satisfies the conditions, and let $p(t) = f(t) + \psi(t)$ be any other function, continuous, with continuous first and second derivatives and such that $p(t_0) = f(t_0) = p_0$, $p(t_1) = f(t_1) = p_1$. Then $\psi(t)$ will be continuous, with its first and second derivatives, and $\psi(t_0) = \psi(t_1) = 0$.

Write

$$\xi(t) = f(t) + x\psi(t)$$

whence

$$\frac{d\xi}{dt} = \xi'(t) = f'(t) + x\psi'(t),$$

x being an arbitrary variable, $0 \leq x \leq 1$. Consider the integral

$$\Pi(x) = \int_{t_0}^{t_1} \pi(\xi, \xi')dt, \tag{5}$$

which for a given arbitrary $\psi(t)$ or $p(t)$ is a function of the single variable x.

On account of the explicit form of $\pi(\xi, \xi')$ it is seen at once, by carrying out the differentiation, that, given ψ(t), the quantity $d\Pi/dx$ exists for every value of x. In fact

$$\frac{d\Pi}{dx} = \int_{t_0}^{t_1} \left\{ \frac{\partial \pi}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \pi}{\partial \xi'}\frac{\partial \xi'}{\partial x} \right\} dt = \int_{t_0}^{t_1} \left\{ \frac{\partial \pi}{\partial \xi}\psi(t) + \frac{\partial \pi}{\partial \xi'}\psi'(t) \right\} dt. \tag{6}$$

Hence it is necessary, in order that $x = 0$ shall correspond to the maximum of Π(*i.e.*, that $f(t)$ shall be the desired function), that

$$\frac{d\Pi}{dx} = 0 \text{ when } x = 0. \tag{a}$$

It is sufficient for a maximum if, no matter what $\psi(t)$ has been given, subject to the enunciated conditions, we have $\Pi(1) < \Pi(0)$; for in this case we shall have

$$\int_{t_0}^{t_1} \pi(p, p')dt < \int_{t_0}^{t_1} \pi(f, f')dt$$

if $p(t)$ is different from $f(t)$. But by the law of the mean:

$$\Pi(1) - \Pi(0) = \left(\frac{d\Pi(x)}{dx}\right)_{x = \bar{x}}$$

for some $\bar{x}$, $0 < \bar{x} < 1$. Hence it is sufficient for a maximum, if given $\psi(t)$ arbitrarily in its domain we have

$$\frac{d\Pi(x)}{dx} < 0 \text{ for all } x,\ 0 < x < 1. \tag{b}$$

It happens that the conditions (a), (b) yield the solution of our problem.

In the expression for $d\Pi/dx$, equation (6), we can perform on the term involving $\psi'(t)$ an integration by parts and write

$$\frac{d\Pi}{dx} = \left[\frac{\partial \pi}{\partial \xi'}\psi(t)\right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left\{ \frac{\partial \pi}{\partial \xi} - \frac{d}{dt}\frac{\partial \pi}{\partial \xi'} \right\} \psi(t)dt, \tag{6.1}$$

in which the first term of the second member vanishes, since $\psi(t_0) = \psi(t_1) = 0$. By a short calculation

$$\frac{\partial\pi}{\partial\xi} = 2a\xi(1 - Aa) + (b - 2Aab - Ba) + h\xi'(1 - 2Aa)$$

$$\frac{\partial\pi}{\partial\xi'} = h\xi(1 - 2Aa) - h(2Ab + B) - 2Ah^2\xi'$$

$$\frac{d}{dt}\left(\frac{\partial\pi}{\partial\xi'}\right) = h\xi'(1 - 2Aa) - 2Ah^2\xi''$$

and therefore

$$\frac{d\Pi}{dx} = \int_{t_0}^{t_1}\{2Ah^2(f''(t) + x\psi''(t)) + 2a(1 - Aa)(f(t) + x\psi(t)) + (b - 2Aab - Ba)\}\psi(t)dt. \quad (7)$$

78. The Conditions (*a*) **and** (*b*).—If (*a*) is to be satisfied with $\psi(t)$ arbitrary, except for the conditions imposed on it, we may

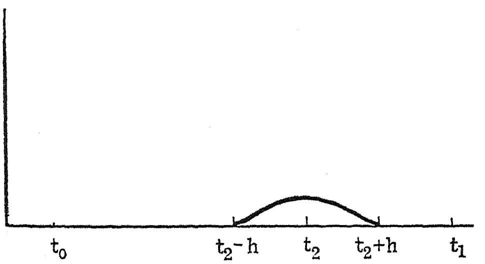

FIG. 27.

deduce from equation (7), after putting $x = 0$, the vanishing of the multiplier of $\psi(t)$ in the integral itself. Hence

$$2Ah^2f''(t) + 2a(1 - Aa)f(t) + (b - 2Aab - Ba) = 0. \quad (8)$$

For if this quantity fails to vanish somewhere and is, say, positive at some value t_2, it will continue to be positive throughout some small interval $(t_2 - h, t_2 + h)$. We can then take $\psi(t)$, vanishing at t_0 and t_1 and continuous with its first derivative, and so that, as in the accompanying figure, it is zero outside the small interval $(t_2 - h,\ t_2 + h)$ and positive in that interval. But for this choice of $\psi(t)$ we have

$$\left[\frac{d\Pi}{dx}\right]_{x=0} = \int_{t_2-h}^{t_2+h}\{\ \}\,\psi(t)dt > 0,$$

where { } stands for the left hand member of (8). But this contradicts condition (a). In this demonstration we have tacitly assumed that the left-hand member of (8) failed to be zero somewhere *between* t_0 and t_1, and in this way obtained a contradiction. But if the expression fails to be zero at either end of the interval, say at t_0, there will be a t_2 near t_0 at which the expression fails to vanish, and therefore the same argument applies. Accordingly the equation (8) must be satisfied identically, $t_0 \leq t \leq t_1$.

In other words, for (a) to be satisfied, $f(t)$ must be a solution of this equation, if there is one, which takes on the values p_0 and p_1 at t_0 and t_1 respectively. We shall see shortly that there is one and only one such solution.

As to the condition (b), if (8) is satisfied, the quantity $d\Pi/dx$ for all x, $0 < x < 1$, reduces to

$$\frac{d\Pi}{dx} = x\int_{t_0}^{t_1}\{2Ah^2\psi''(t) + 2a(1 - Aa)\psi(t)\}\psi(t)dt$$

or

$$\frac{d\Pi}{dx} = x\Big[2Ah^2\psi'(t)\psi(t)\Big]_{t_0}^{t_1} + x\int_{t_0}^{t_1}\{-2Ah^2(\psi'(t))^2 + 2a(1 - Aa)(\psi(t))^2\}dt,$$

when we retrace the integration by parts on the first term in the above integral. In the last expression the term outside the integral is again zero, since $\psi(t_0) = \psi(t_1) = 0$, and the integral term is negative, on account of the inequalities $A > 0$, $a < 0$, unless $\psi(t) \equiv 0$.

Condition (b), as well as condition (a), is therefore satisfied if $f(t)$ is the desired solution of (8). This equation is solvable in terms of exponential or hyperbolic functions. Thus if we introduce the constants

$$f_0 = \frac{b - 2Aab - Ba}{-2a(1 - Aa)},\quad m^2 = \frac{-a(1 - Aa)}{Ah^2},$$

the equation may be written in the form

$$f'' - m^2 f = -m^2 f_0, \tag{9}$$

and the general solution of the equation is seen by trial to be

$$f(t) = f_0 + C_1 e^{m(t-t_0)} + C_2 e^{-m(t-t_0)}, \tag{10}$$

with C_1 and C_2 arbitrary constants and m the positive root of m^2. If we let $r_0 = p_0 - f_0$, $r_1 = p_1 - f_0$, substitution in this formula of the values $t = t_0$ and $t = t_1$ respectively gives us

$$r_0 = C_1 + C_2$$
$$r_1 = C_1 e^{m(t_1 - t_0)} + C_2 e^{-m(t_1 - t_0)}$$

whence

$$C_1 = \frac{r_0 - r_1 e^{m(t_1 - t_0)}}{1 - e^{2m(t_1 - t_0)}}$$
$$C_2 = \frac{r_0 - r_1 e^{-m(t_1 - t_0)}}{1 - e^{-2m(t_1 - t_0)}} \tag{10.1}$$

There is therefore always a unique solution of the problem proposed.

79. Discussion of Solution.—A particular solution of (9) is seen, by putting $C_1 = 0 = C_2$, to be $f(t) = f_0$. It is interesting to note that this price is just the monopoly equilibrium price of Chapter I, obtained when the equation of demand did not involve h. It continues to be a solution of the present problem, with $h \neq 0$, if the end values of p_0 and p_1 are taken the same and equal to f_0. Moreover the formulae (10), (10.1) show that the only solutions which remain finite as t_1 becomes infinite are those for which $C_1 = 0$ and which therefore approach f_0 asymptotically, *viz.*,

$$f(t) = f_0 + r_0 e^{-m(t - t_0)}. \tag{11}$$

No solution of (9) which is not identically equal to f_0 can take on the value f_0 more than once; in fact, as is seen from (10), this will happen only for the value of t, if that value is real and in the interval (t_0, t_1), for which

$$e^{2m(t - t_0)} = -\frac{C_2}{C_1}.$$

The calculation of the slope df/dt gives an accurate picture of the graph of $f(t)$ as a function of t. From (10),

$$\frac{df}{dt} = m\left[C_1 e^{m(t - t_0)} - C_2 e^{-m(t - t_0)}\right].$$

At the time t_0 this slope has the value

$$f_0' = m(C_1 - C_2),$$

and the slope will vanish for the single value of t (if that value is real and in the interval (t_0, t_1) for which

$$e^{2m(t - t_0)} = \frac{C_2}{C_1}.$$

By calculating the values of $C_1 - C_2$ and C_2/C_1 in terms of r_0 and r_1 the reader will find it interesting to verify the following facts with respect to the graph of price against time.

If r_0 and r_1 have opposite signs, the graph crosses the line $p = f_0$ once and has no horizontal tangent in the interval, the price continuously decreasing or increasing, as the case may be, from p_0 to p_1; if r_0 and r_1 have the same sign, the graph fails to cross the line $p = f_0$ at all, and has one and only one horizontal tangent provided the interval is large enough,—*i.e.*, provided

$$\cosh m(t_1 - t_0) \geq \frac{r_0}{r_1} \geq \frac{1}{\cosh m(t_1 - t_0)}$$

Otherwise there is no horizontal tangent. Whether there is a maximum or a minimum is decided in any case of the existence

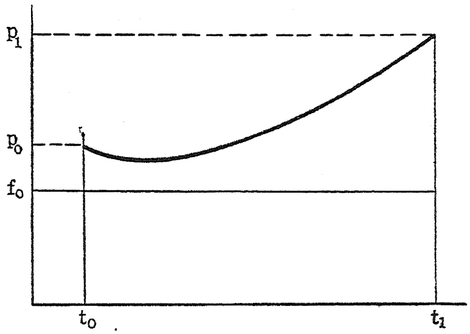

FIG. 28.

of a horizontal tangent (since the second derivative does not vanish) by the slope of the graph at $t = t_0$, and this is readily seen to have the same algebraic sign as the quantity

$$r_1 - r_0 \cosh m(t_1 - t_0)$$

But a comparison of this quantity with the previous inequality shows that when r_0 and r_1 are both positive the graph can have only a minimum (see accompanying figure), and when r_0 and r_1 are both negative only a maximum; in other words, when r_0 and r_1 have the same sign, the graph is contained between f_0 and one of the values p_0 or p_1.

These results are all independent of the algebraic sign of h in (2). It is merely a question of which end of the graph contributes most of the profit or least of the loss—since it is not essential that Π itself shall be positive.

80. An Extension of the Domain of $p(t)$.—Obviously it is usually not possible to keep the slope of the price curve continuous at the times $t = t_0$ and $t = t_1$, as we see in the case when we change from one price p_0, which has been constant, to a second price p_1, which will be constant; it is therefore unreasonable to restrict ourselves to the consideration of curves whose slopes remain continuous within the interval (t_0, t_1). It is interesting to note that the solution already obtained is valid even when this restriction has been removed from the class of functions with which we deal in forming Π.

In fact, let $f(t)$ be the function already found, and $p(t)$ any other function taking on the same initial and final values, continuous and with a derivative which remains bounded but may have discontinuities at a finite number of places. Then $\psi(t) = p(t) - f(t)$ and $\xi(t) = f(t) + x\psi(t)$ will be functions of the same sort. It remains to show that condition (*b*) will still be satisfied.

If in the evaluation of the second member of (6) we separate out those terms which do not involve x, they will cancel among themselves, for they form precisely the quantity $(d\Pi/dx)_{x=0}$ which involves the part $f(t)$ of $\xi(t)$ and vanishes since $f(t)$ satisfies (9). Hence

$$\frac{d\Pi}{dx} = x\int_{t_0}^{t_1}[2a(1 - Aa)\{\psi(t)\}^2 - 2Ah^2\{\psi'(t)\}^2 + 2h(1 - 2Aa)\psi(t)\psi'(t)]dt.$$

But

$$2\int_{t_0}^{t_1}\psi(t)\psi'(t)dt = \int_{t_0}^{t_1}\frac{d}{dt}\{\psi(t)\}^2dt = \{\psi(t)\}^2]_{t_0}^{t_1} = 0,$$

since $\psi(t_0) = \psi(t_1) = 0$, and the rest of $d\Pi/dx$ is essentially negative since $a < 0, A > 0$. The point is thus proved.

81. An Extension of the Problem.—We consider now the problem when merely the value p_0 is given and we try to make Π a maximum over the interval of time (t_0, t_1) without specifying the final value p_1. Assuming that there will be an optimum value p_1 let us see how we can determine it. Whatever p_1 may be, the function which will then make Π a maximum will be, as we have seen, the solution of (9) which takes on the initial value p_0 and the final value p_1; hence we need compare the values of Π merely for functions $p(t)$ which are solutions of (9). Such functions $p(t)$ are called extremals.

The family of extremals such that $p(t_0) = p_0 = f_0 + r_0$ may be written in the form

$$p(t) = f_0 + r_0 \cosh m(t - t_0) + C \sinh m(t - t_0) \qquad (12)$$

since the hyperbolic sines and cosines are merely linear combinations of exponential functions

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x})$$
$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}),$$

and the function (12) evidently is equal to $p_0 = f_0 + r_0$ when $t = t_0$. We shall find what value of C will make $d\Pi/dC = 0$.

We have from (4)

$$\begin{aligned}\frac{d\Pi}{dC} &= \int_{t_0}^{t_1} \left\{ \frac{\partial \pi}{\partial p}\frac{\partial p}{\partial C} + \frac{\partial \pi}{\partial p'}\frac{\partial p'}{\partial C} \right\} dt \\ &= \left[\frac{\partial \pi}{\partial p'}\frac{\partial p}{\partial C} \right]_{t_0}^{t_1} + \int_{t_0}^{t_1} \left(\frac{\partial \pi}{\partial p} - \frac{d}{dt}\frac{\partial \pi}{\partial p'} \right) \frac{\partial p}{\partial C}\, dt \\ &= \left(\frac{\partial \pi}{\partial p'}\frac{\partial p}{\partial C} \right)_{t = t_1} = \left(\frac{\partial \pi}{\partial p'} \right)_{t = t_1} \sinh m(t_1 - t_0)\end{aligned}$$

In fact, the integral term vanishes, in the second equation just given, since the quantity

$$\frac{\partial \pi}{\partial p} - \frac{d}{dt}\frac{\partial \pi}{\partial p'}$$

is nothing else but the quantity in $\{\}$ in the expression (7). Moreover the expression outside the integral sign vanishes at $t = t_0$ since $\partial p/\partial C = \sinh m(t - t_0)$ vanishes for that value of t. But

$$\frac{\partial \pi}{\partial p'} = hp(1 - 2Aa) - h(2Ab + B) - 2Ah^2p'$$

and therefore when we substitute for p and p' their values, from (12), we have

$$\begin{aligned}\frac{d\Pi}{dC} = h \sinh m(t_1 - t_0)[(f_0 + r_1 \cosh m(t_1 - t_0))(1 - 2Aa) \\ - (2Ab + B) - 2Ahr_1m \sinh m(t_1 - t_0) \\ + C\{(1 - 2Aa) \sinh m(t_1 - t_0) - 2Ahm \cosh m(t_1 - t_0)\}] \quad (13)\end{aligned}$$

Since $d\Pi/dC$ is a linear function of C it can vanish for only one value of C; *there is therefore only one possible optimum value of C and therefore only one possible optimum value of p_1.* There will always be a value of C which will make $d\Pi/dC = 0$ unless the coefficient of C in (13) happens to be zero. Let this value be C_1.

The question remains as to whether the value C_1, which makes (13) vanish, furnishes a maximum for Π or some other kind of a horizontal tangent on the graph of Π against C. But this question also can be answered definitely; for we shall have a maximum if $d^2\Pi/dC^2 < 0$. From (13) we have

$$\frac{d^2\Pi}{dC^2} = \sinh m(t_1 - t_0)\{h(1 - 2Aa) \sinh m(t_1 - t_0) \\ -2Ah^2m \cosh m(t_1 - t_0)\}$$

for all values of C. This quantity is obviously < 0 if $h < 0$. Moreover $m = \sqrt{m^2} = \sqrt{-a(1-Aa)}/(\sqrt{A}\,|h|,)$, so that if $h > 0$,

$$\frac{d^2\Pi}{dC^2} = 2h\sqrt{-Aa(1 - Aa)} \sinh m(t_1 - t_0) \cosh m(t_1 - t_0) \\ \left\{\frac{(1 - 2Aa)}{2\sqrt{-Aa(1 - Aa)}} \tanh m(t_1 - t_0) - 1\right\} \quad (14)$$

Now $\dfrac{(1 - 2Aa)}{2\sqrt{-Aa(1 - Aa)}}$ is necessarily > 1 and as a quantity α increases from 0 to ∞ the function $\tanh \alpha$ increases continuously from 0 to 1. Hence if $h > 0$ there will always be a value T such that if $t_1 - t_0 < T$ the bracket in the above expression will be < 0 and if $t_1 - t_0 > T$ this bracket will be > 0, so that C_1 will be an optimum value if $t_1 - t_0 < T$ and will not be if $t_1 - t_0 > T$. The result of this section can accordingly be expressed as follows:

If $h < 0$ there is a single $p(t)$ which takes on the initial value p_0 at $t = t_0$ and makes Π a maximum for the interval (t_0, t_1). This function will be the extremal (12) *for which C is the value C_1 which makes* (13) *vanish* [*compare Exercise* 1, Sec. **82**]. *The same is true for $h > 0$ provided $t_1 - t_0 < T$, but is not true if $t_1 - t_0 > T$.*

Since from (14), T will depend on m and therefore on h, as well as on A and a, we see that T will be a function of h, A, and a.

If h is positive then and if $t_1 - t_0$ is greater than a certain interval of time T the profit can be made as great as we please by taking the proper extremal; for Π is a quadratic function of C and can therefore have only one maximum or minimum. But for very large changes of price it is not reasonable to assume the law (2) to remain valid; and the conclusion is, therefore, that for long intervals of time it may be profitable for the producer to form a price curve which leads to the boundary of the region

where the hypothesis (2) remains applicable, and where a different law then comes into effect. Or we can imagine as a further refinement to (2) a law in which the a, h are not constants but functions of p, p'; for example, we might let h take the constant value h_1 or the constant value h_2 according as p' is positive or negative respectively.

82. General Exercises.

1. Show that if $h > 0$ and $t_1 - t_0 = T$ in Sec. **81** the equation (13) does not determine C_1.

2. If b is a function of t, $b = b(t)$, what is the effect on the analysis of Sec. **77** and the equation corresponding to (9)?

3. What will be the analysis of the problems of competition and cooperation which takes the place of Secs. **76, 77**?[1]

[1] For a study of competition, consult Roos, C. F., A Mathematical Theory of Competition, *Amer. Jour. Math.*, vol. 47, pp. 163–175, 1925.

CHAPTER XV

PROBLEMS OF SIGNIFICANT TYPES

83. Production Lag.—The reader will notice that the problem treated in the previous chapter admits of considerable generalization, and will find a general formulation of that character in Appendix II. The simple monopoly problem, as we have considered it, depends upon three relations: (a), that between the amount produced u and the cost of it q; (b), that between the demand y and the price p; and (c), the relation between the amount produced and the demand. For any of the particular functions chosen we may of course substitute more general functions, as in particular, in (b), we may replace

$$y = ap + b + h\frac{dp}{dt}$$

by

$$y = ap + b + h\frac{dp}{dt} + k\frac{d^2p}{dt^2}$$

More significant however is the generalization which we obtain when, with an eye to practical needs, we substitute for a given relation another of different mathematical character. Such extensions of the problem are given in the present chapter. It is hoped that the reader will devise others for his own amusement, and with reference to practical and theoretical interests.

We consider first a problem where the relation between production and demand, $y = u$, is replaced by another. We shall assume that there is a lag in the production, so that it is governed not by the present demand but by the demand at a certain time before; we take then

$$u(t) = y(t - T) \tag{1}$$

where T is a given constant interval of time. For instance if T is one month, the equation (1) says that the production per unit time at time t is equal to the demand per unit time one month earlier. We take again

$$\begin{aligned} y(t) &= ap(t) + b \\ q &= A(u(t))^2 + Bu(t) + C \end{aligned} \tag{1.1}$$

and try to find a condition on $p(t)$ in order that the profit Π over the interval (t_0, t_1):

$$\Pi = \int_{t_0}^{t_1} \{p(t)y(t) - A(u(t))^2 - Bu(t) - C\}dt \tag{2}$$

shall be a maximum. We assume that the price $p(t)$ is known up to the time $t = t_0$, and that the producer is not concerned about what happens after $t = t_1$; he may be going out of business.

From (1), (1.1), we have

$$u(t) = y(t - T) = ap(t - T) + b$$

$$\Pi = \int_{t_0}^{t_1} \{a(p(t))^2 + bp(t) - A(ap(t - T) + b)^2 - B(ap(t - T) + b) - C\}dt,$$

and, as in Chapter XIV, we reduce Π to a function of a single variable x by replacing $p(t)$ by $p(t) + x\psi(t)$, and $p(t - T)$ by $p(t - T) + x\psi(t - T)$. Then we may see if it is possible for $p(t)$ to furnish a maximum for Π by making

$$\frac{d\Pi}{dx} = 0 \text{ for } x = 0.$$

But since $p(t - T)$ is known when $t < t_0 + T$ we must put $\psi(t - T) = 0$ when $t < t_0 + T$ in the part of Π which refers to the cost function. With these substitutions, we find immediately,

$$\frac{d\Pi}{dx} = \int_{t_0}^{t_1} [2a\{p(t) + x\psi(t)\} + b]\psi(t)dt - \int_{t_0 + T}^{t_1} [2Aa\,(a\{p(t - T) + x\psi(t - T)\} + b) + Ba]\psi(t - T)dt,$$

which gives us

$$0 = \left(\frac{d\Pi}{dx}\right)_{x=0} = \int_{t_0}^{t_1} (2ap(t) + b)\psi(t)dt - \int_{t_0 + T}^{t_1} [2Aa(ap(t - T) + b) + Ba]\psi(t - T)dt.$$

If in the first integral we write $t = \tau$ and in the second $t - T = \tau$, where τ is a new variable of integration, this equation reduces to the following:

$$0 = \int_{t_0}^{t_1} (2ap(\tau) + b)\psi(\tau)d\tau - \int_{t_0}^{t_1 - T} [2Aa(ap(\tau) + b) + Ba]\psi(\tau)dt$$

and if we collect terms, we have

$$0 = \int_{t_0}^{t_1 - T} [2a(1 - Aa)p(\tau) + b - 2Aab - Ba]\psi(\tau)d\tau$$
$$+ \int_{t_1 - T}^{t_1} (2ap(\tau) + b)\psi(\tau)d\tau. \quad (3)$$

But since $\psi(\tau)$ is arbitrary from $\tau = t_0$ to $\tau = t_1$, both of these integrands must vanish. In other words

$$\begin{aligned} p(\tau) &= \frac{b - 2Aab - Ba}{-2a(1 - Aa)}, \text{ for } t_0 < \tau < t_1 - T \\ &= \frac{b}{-2a}, \text{ for } t_1 - T < \tau < t_1, \end{aligned} \quad (4)$$

so that we get two different prices for the two parts of the interval of time from t_0 to t_1 (assuming that $T < t_1 - t_0$).

Since

$$\frac{b - 2Aab - Ba}{-2a(1 - Aa)} = \frac{b}{-2a} + \frac{Ab + B}{2(1 - Aa)},$$

of which the second term of the right hand member is positive under the conditions $A > 0$, $B > 0$, $b > 0$, $a < 0$, the first of the two prices is greater than the second. *For maximum* Π *the price is held at the familiar monopoly price until the time* $t_1 - T$, *and then for the rest of the time, until* t_1, *is dropped to the value* $p = -b/2a$. For the production $u(t)$ which produces this price, we have

$$u(t) = ap(t - T) + b,$$

for t in the time interval $(t_0, t_0 + T)$, since $p(t)$ is known up to the time $t = t_0$; and

$$u(t) = \frac{b + Ba}{2(1 - Aa)},$$

for t in the time interval $(t_0 + T, t_1)$, since in the interval $(t_0 + T, t_1)$ the $p(t - T)$ is given by the first of the formulae (4). The demand during the time $(t_1 - T, t_1)$ is seen, from the second of the formulae (4), to be $b/2$.

84. Discontinuous Production.—As a second problem let us change the relation between u and q by assuming that the cost is still a quadratic function of the amount produced, but that it is all produced at the time t_0 and sold gradually thereafter. This kind of discontinuous production is usually periodic in that so much is manufactured every so often—as with crops—but we shall consider merely one period. We take

$$y(t) = ap(t) + b$$

and the total amount produced U equal to the total demand

$$U = \int_{t_0}^{t_1} y(t)dt$$

the cost being, say,

$$AU^2 + BU + C(t_1 - t_0) + C'.$$

For Π we express all the functions in terms of $y(t)$ and obtain

$$\Pi = \int_{t_0}^{t_1} \frac{(y(t))^2 - by(t)}{a} dt - A\left[\int_{t_0}^{t_1} y(t)dt\right]^2 - B\int_{t_0}^{t_1} y(t)dt \\ -C(t_1 - t_0) - C'. \quad (5)$$

Here again, in order to find the function $y(t)$ which makes Π a maximum we replace $y(t)$ by $y(t) + x\psi(t)$ and make use of the condition

$$\left(\frac{d\Pi}{dx}\right)_{x=0} = 0.$$

Since

$$\left[\int_{t_0}^{t_1}(y(t) + x\psi(t))dt\right]^2 = \left[\int_{t_0}^{t_1} y(t)dt\right]^2 \\ + 2x\left[\int_{t_0}^{t_1} y(t)dt\right]\left[\int_{t_0}^{t_1} \psi(t)dt\right] + x^2\left[\int_{t_0}^{t_1} \psi(t)dt\right]^2$$

we have

$$0 = \left(\frac{d\Pi}{dx}\right)_{x=0} = \int_{t_0}^{t_1}\left(\frac{2y(t) - b}{a} - B\right)\psi(t)dt \\ -2A\left(\int_0^{t_1} y(t)dt\right)\left(\int_{t_0}^{t_1} \psi(t)dt\right).$$

This may be written in the form

$$\int_{t_0}^{t_1}\left[\frac{2y(t) - b}{a} - B - 2A\int_{t_0}^{t_1} y(\tau)d\tau\right]\psi(t)dt = 0,$$

since

$$\int_{t_0}^{t_1} y(t)dt = \int_{t_0}^{t_1} y(\tau)d\tau.$$

But again the $\psi(t)$ is arbitrary, and therefore $y(t)$ must satisfy the equation

$$2y(t) - b - Ba - 2Aa\int_{t_0}^{t_1} y(\tau)d\tau = 0. \quad (6)$$

This equation may be solved at once. For $\int_{t_0}^{t_1} y(\tau)d\tau$ is some constant m, so that

$$y(t) = \tfrac{1}{2}(b + Ba + 2Aam). \quad (7)$$

In order to find the value m we need merely integrate (6) from t_0 to t_1, and we obtain

$$2m - (b + Ba)(t_1 - t_0) - 2Aam(t_1 - t_0) = 0$$

or

$$m = \frac{(b + Ba)(t_1 - t_0)}{2\{1 - Aa(t_1 - t_0)\}}. \tag{7.1}$$

The function $y(t)$ is thus given by (7) with (7.1), and from $y(t)$ the other functions may be written down.

85. A Demand Law with an Integral.—As a final problem we consider a demand law of the form

$$p(t) = ry(t) + s + \int_{t_0}^{t} k(t, \tau)y(\tau)d\tau, \tag{8}$$

which says that the price at time t depends linearly on the demand at time t, and also linearly on all the previous demands from time t_0 to t, the contribution during $\Delta\tau$ at any time τ between t_0 and t, being approximately $k(t,\tau)\ y(\tau)\Delta\tau$.[1] In order to simplify matters we may assume $k(t, \tau)$ to be a function of $t - \tau$ only, so that the contribution of a demand $y(\tau)$ depends merely on the interval of time by which τ precedes t; we shall in fact take the simplest possible law of this kind and write $k(t, \tau)$ as a constant k times $(t - \tau)$, so that (8) takes the form

$$p(t) = ry(t) + s + k \int_{t_0}^{t} (t - \tau)y(\tau)d\tau. \tag{8.1}$$

In order to have a definite situation we assume $r < 0$, $k > 0$.

We shall assume, as before,

$$y(t) = u(t);\ q = Au^2 + Bu + C,$$

with A, B, C all positive. We write down the condition $d\Pi/dx = 0$ for $x = 0$, introducing the variable x as in the earlier problems.

We have

$$\Pi = \int_{t_0}^{t_1} \Big[r(y(t))^2 + sy(t) + ky(t) \int_{t_0}^{t} (t - \tau)y(\tau)d\tau$$
$$- A(y(t))^2 - By(t) - C \Big] dt.$$

[1] C. F. ROOS applied this demand law to the theory of competition, in his thesis towards the degree of M. A., at the Rice Institute (Roos, *Amer. Jour. Math.*, vol. 47, July, 1925). Laws of this kind were first discussed by VOLTERRA, in relation to problems in physics (resumed in his book, "*Fonctions de lignes*," Paris, 1913).

If we replace $y(t)$ by $y(t) + x\psi(t)$, $y(\tau)$ by $y(\tau) + x\psi(\tau)$, differentiate with respect to x and let x approach 0, we obtain

$$0 = \left(\frac{d\Pi}{dx}\right)_{x=0} = \int_{t_0}^{t_1}\Big[2ry(t) + s - 2Ay(t) - B$$
$$+ k\int_{t_0}^{t}(t-\tau)y(\tau)\,d\tau\Big]\psi(t)dt + \int_{t_0}^{t_1} ky(t)\left(\int_{t_0}^{t}(t-\tau)\psi(\tau)d\tau\right)dt.$$

This last integral may however be interpreted as a double or area integral

$$\int ky(t)(t-\tau)\psi(\tau)d\sigma$$

extended over the triangular area illustrated in the accompanying figure; for in performing this double integration over the area indicated, if we integrate first with regard to τ, holding t fixed, we

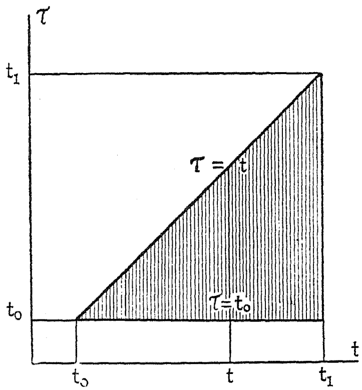

FIG. 29.

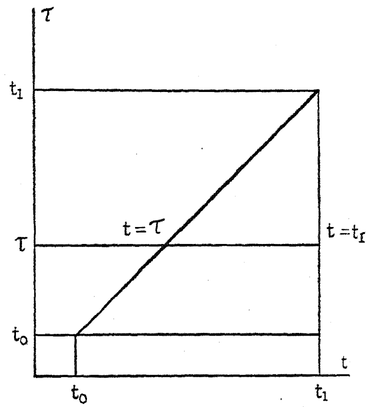

FIG. 30.

make the summation precisely over values of τ from $\tau = t_0$ to $\tau = t$; and having performed this integration we then integrate with respect to t from t_0 to t_1. But we may also perform this same integration over the shaded area by carrying out first the integration with respect to t. In this latter case we must hold τ constant, and as we carry out the integration with respect to t we shall travel from $t = \tau$ to $t = t_1$ (see Fig. 30). We then integrate with respect to τ from $\tau = t_0$ to $\tau = t_1$.

We have therefore the identity,[1] which is of frequent use in problems of this sort and may be stated no matter what function of t and τ the integrand happens to be,

$$\int_{t_0}^{t_1}\left(\int_{t_0}^{t} ky(t)(t-\tau)\psi(\tau)d\tau\right)dt = \int_{t_0}^{t_1}\left(\int_{\tau}^{t_1} ky(t)(t-\tau)\psi(\tau)dt\right)d\tau.$$

In this last integral t and τ are merely variables of integration, which are not involved in the final result, and we may rename them in any way we please. In particular we may substitute τ for t and t for τ and write the integral also as the following

$$\int_{t_0}^{t_1}\left(\int_{t}^{t_1} ky(\tau)(\tau - t)\psi(t)d\tau\right)dt.$$

The $\psi(t)$ may however be removed from the inside integration, which is with respect to τ, and the last integral of (9) becomes thus finally

$$\int_{t_0}^{t_1}\left(\int_{t}^{t_1} k(\tau - t)y(\tau)d\tau\right)\psi(t)dt.$$

Hence (9) becomes

$$0 = \int_{t_0}^{t_1}\left[2(r-A)y(t) + s - B + k\int_{t_0}^{t}(t-\tau)y(\tau)d\tau + k\int_{t}^{t_1}(\tau - t)y(\tau)d\tau\right]\psi(t)dt,$$

and since $\psi(t)$ is arbitrary we must have the [] identically zero. Accordingly

$$2(r-A)y(t) + s - B + k\int_{t_0}^{t}(t-\tau)y(\tau)d\tau + k\int_{t}^{t_1}(\tau - t)y(\tau)d\tau = 0.$$

Since $t - \tau$ is ≥ 0 in the first of the integrals and $\tau - t$ is ≥ 0 in the second, we may write finally

$$2(r-A)y(t) + s - B + k\int_{t_0}^{t_1}|t-\tau|y(\tau)d\tau = 0, \qquad (10)$$

as an equation to determine $y(t)$.

For convenience in solving this equation we write it in the form

$$2y(t) + \alpha - \beta^2\int_{t_0}^{t}(t-\tau)y(\tau)d\tau - \beta^2\int_{t}^{t_1}(\tau - t)y(\tau)d\tau = 0 \qquad (11)$$

where

$$\alpha = \frac{s-B}{r-A}, \quad \beta^2 = \frac{k}{A-r}, \qquad (11.1)$$

[1] Compare OSGOOD, "Differential and Integral Calculus," chap. XVIII, Secs. **2, 4** and Exercises 6(*b*), 7, p. 376, New York, 1922.

the β^2 being positive if $k > 0$, $r < 0$. If we differentiate the equation preceding (10) we obtain

$$2y'(t) - \beta^2\int_{t_0}^{t} y(\tau)d\tau + \beta^2\int_{t}^{t_1} y(\tau)d\tau = 0,$$

and if we differentiate again,

$$y''(t) - \beta^2 y(t) = 0.$$

But the general solution of this equation is known to be

$$y(t) = C_1e^{\beta t} + C_2e^{-\beta t} \tag{11.1}$$

It remains then merely to determine the constants C_1 and C_2 by substitution into (11).

We have, by carrying out the integration term by term,

$$\begin{aligned} &2C_1e^{\beta t} + 2C_2e^{-\beta t} + \alpha \\ &\quad + C_1\beta t\{-2e^{\beta t} + e^{\beta t_0} + e^{\beta t_1}\} + C_2\beta t\{2e^{-\beta t} - e^{-\beta t_0} - e^{-\beta t_1}\} \\ &\quad + C_1\{2\beta te^{\beta t} - 2e^{\beta t} - \beta t_0e^{\beta t_0} - \beta t_1e^{\beta t_1} + e^{\beta t_0} + e^{\beta t_1}\} \\ &\quad + C_2\{-2\beta te^{-\beta t} - 2e^{-\beta t} + \beta t_0e^{-\beta t_0} + \beta t_1e^{-\beta t_1} \\ &\qquad\qquad + e^{-\beta t_0} + e^{-\beta t_1}\} = 0, \end{aligned}$$

or

$$\begin{aligned} &\beta t[C_1(e^{\beta t_0} + e^{\beta t_1}) - C_2(e^{-\beta t_0} + e^{-\beta t_1})] + \alpha \\ &+ C_1\{e^{\beta t_0} + e^{\beta t_1} - \beta t_0e^{\beta t_0} - \beta t_1e^{\beta t_1}\} + C_2\{e^{-\beta t_0} + e^{-\beta t_1} \\ &\qquad + \beta t_0e^{-\beta t_0} + \beta t_1e^{-\beta t_1}\} = 0. \end{aligned}$$

But in this last equation the terms that involve βt cannot cancel against those that do not. The following two equations must therefore be satisfied

$$\begin{aligned} &C_1(e^{\beta t_0} + e^{\beta t_1}) - C_2(e^{-\beta t_0} + e^{-\beta t_1}) = 0 \\ &C_1(e^{\beta t_0} + e^{\beta t_1} - \beta t_0e^{\beta t_0} - \beta t_1e^{\beta t_1}) + C_2(e^{-\beta t_0} + e^{-\beta t_1} \\ &\qquad + \beta t_0e^{-\beta t_0} + \beta t_1e^{-\beta t_1}) = -\alpha. \end{aligned} \tag{12}$$

Conversely if these equations are satisfied by C_1 and C_2, the function $y(t)$, given by (11.1), will satisfy (11).

The equations (12) constitute two linear equations in C_1 and C_2, and therefore, in general, may be solved for C_1 and C_2. By carrying out the solution it will be seen that these equations may be solved unless $\beta(t_1 - t_0)$ happens to have a particular value γ determined by the equation

$$4 + e^{\gamma}(2 - \gamma) + e^{-\gamma}(2 + \gamma) = 0$$

or

$$4 + 2(e^{\gamma} + e^{-\gamma}) - \gamma(e^{\gamma} - e^{-\gamma}) = 0 \tag{12.1}$$

The left-hand member of this equation can be shown to be a continually decreasing function of γ, $\gamma > 0$, by calculating its derivative with respect to γ. Hence there is not more than one positive value of γ which will satisfy (12.1). On the other hand since the left-hand member of (12.1) is > 0 for small γ and < 0 for large γ there is such a value. Accordingly *there is one and only one value of* $\beta(t_1 - t_0)$ *for which* (11) *cannot be solved uniquely for* $y(t)$.

86. General Mathematical Methods.—The method which has been utilized in the last two chapters has been to investigate the variations in a quantity Π due to variations in the functions $p(t)$, $y(t)$, $u(t)$, etc. This general method has been extensively used, especially in applications to physics, for a century, and has generated a branch of mathematics which is commonly known as the calculus of variations. In the treatment just given only elementary methods have been utilized, since it has always been possible to arrive at a condition for the function desired and to investigate the maximal character of the resulting situation by reducing the problem to one in the maxima and minima of functions of a single variable x, the Π being a quadratic function of that variable. In problems where several producers or products are involved and the functions are equally simple, the solution becomes possible by introducing several variables analogous to the x. But such methods will not always give an adequate analysis; and the reader who desires to be well equipped for this sort of study will do well to familiarize himself to some extent with the branch of mathematics just mentioned.[1]

It will be noticed also that having found the condition or conditions which hold on the function $p(t)$, or whatever function is to be determined, in order that Π may be a maximum—equations (9), Sec. **78**; (6) Sec. **84**; (10) Sec. **85**, etc.—there is still a mathematical process involved in passing from the condition on the function to the function itself. In (9), Sec. **78**, this process consisted in the solution of a simple differential equation; in (6), Sec. **84**, and (10), Sec. **85**, equations in integrals had to be solved; and in (3), Sec. **83**, and earlier in Exercise 2, Sec. **27**,

[1] For the calculus of variations, see OSGOOD "Advanced Calculus," chap. XVII; WOODS, "Advanced Calculus," chap. XIV; BLISS, "Calculus of Variations," Open Court Publishing Co., Chicago, 1925. The last is one of the Carus Mathematical Monographs, and is designed to be readable with the same mathematical preparation as the chapters in advanced calculus just cited in this connection.

Chapter IV, there is a suggestion of analysis in terms of differences rather than differentials or integrals. These processes then also point in the direction of further desirable mathematical equipment.[1]

Rather than embark immediately however upon a profound study of these mathematical disciplines, the student will do well to choose from the brief references given in this section; and then to select some problem in economics which is interesting to him and pursue it wherever it may lead.

87. General Exercises.

1. Find out if the solutions given in Secs. **83** and **84** furnish a maximum value of Π, or some other kind of a horizontal tangent in the graph of $\Pi(x)$. When Π is a quadratic function of x, $\Pi(1)$ will be less than $\Pi(0)$ if $[d^2\Pi(x)/dx^2]$ is < 0, since $[d\Pi/dx]_{x=0} = 0$.

2. Carry out the solution of the problem in Sec. **84** if b is no longer constant, but is a given function of t, say $b(t)$.

3. Verify the remarks with respect to the solutions of equations (12) and (12.1), in the text.

4. Find $d^2\Pi/dx^2$ in the problem of Sec. **85**. Compare with (12.1).

5. It is known that if

$$y(t) = ap(t) + b(t) + a\int_t^t K(t, \tau)p(\tau)d\tau$$

then

$$p(t) = ry(t) + s(t) + r\int_{t_0}^t k(t, \tau)y(\tau)d\tau$$

where $r = 1/a$ and $k(t, \tau)$ depends merely on $K(t, \tau)$. This means that if the demand depends in this way on the price, given from t_0 to t, the price depends on the demand given merely from t_0 to t. Is this result evident a priori?

6. If in (8.1) k is taken to be a negative constant, show that the movement becomes oscillatory.

7. Discuss the problem in the case of monopoly and the differential demand law of Chapter XIV if the condition

$$\int_{t_0}^{t_1} u(t)\, dt = K$$

is added, that is, if the total production during the interval of time (t_0, t_1) is given.

[1] For differential equations, see OSGOOD, *op. cit.*, chap. XIV; WOODS, *op. cit.*, chap. X*ff*. For integral equations, see BOCHER, "An Introduction to the Study of Integral Equations," (Cambridge Tracts in Mathematics and Mathematical Physics No. 10), Cambridge, 1914. A collection of useful exercises will be found in LOVITT, "Linear Integral Equations," New York, 1924.

8. Discuss the same sort of problem if the condition $y(t) = u(t)$ is replaced by the requirement that the total production equals the total demand:

$$\int_{t_0}^{t_1} y(t)dt = \int_{t_0}^{t_1} u(t)dt,$$

where the value of these equal quantities may or may not be given in advance.

(Exercises 7 and 8 require some slight additional reading in the Calculus of Variations.)

APPENDIX I

BIBLIOGRAPHY OF COLLATERAL READING IN ENGLISH

The reader should have at hand, for frequent reference, some text book on the calculus—preferably the one with which he is most familiar,—and, for occasional reference, some Advanced Calculus, such as that of OSGOOD, The Macmillan Company, New York, 1925, or WOODS, Ginn and Company, Boston, 1926. It will be worth while also to read in some one of the well-known treatises on economics such as that of MARSHALL, "Principles of Economics," The Macmillan Company, London, 1920, or PIGOU, "The Economics of Welfare," The Macmillan Company, London, 1924. These and the older classical treatments like those of Ricardo and Adam Smith will suggest many problems for which a mathematical analysis is necessary.

In connection with the various chapters of the book the following references may be given for further reading:

Chapters I, III, V.—COURNOT, "Mathematical Principles of the Theory of Wealth," trans. by Bacon, chaps. I, II, IV–VIII, New York, 1927. EVANS, A Simple Theory of Competition, *Amer. Math. Monthly*, vol. 29, Nov.-Dec., 1922. SCHULTZ, The Statistical Law of Demand, *Jour. Polit. Econ.*, vol. 33, Oct., Dec., 1925.

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Chapter IV.—MOORE, A Moving Equilibrium of Demand and Supply, *Quart. Jour. Econ*, May, 1925. MOORE, A Theory of Economic Oscillations, *Quart. Jour. Econ*, Nov., 1926.

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Chapter VIII.—FISHER, "The Rate of Interest," New York, 1907.

Chapter IX.—FISHER, "The Purchasing Power of Money," New York, 1922. FISHER, The Business Cycle Largely a "Dance of the Dollar," *Quart. Pub. Amer. Statistical Assoc.*, Dec., 1923. LAUGHLIN, "The Principles of Money," New York, 1921. JEVONS, "Money and the Mechanism of Exchange," London, 1892.

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Chapter XII.—FISHER, A Statistical Method for Measuring "Marginal Utility" and Testing the Justice of a Progressive Income Tax, in "Economic Essays in Honor of John Bates Clark," New York, 1927.

Chapter XIV.—EVANS, The Dynamics of Monopoly, *Amer. Math. Monthly*, vol. 31, Feb., 1924. ROOS, A Mathematical Theory of Competition, *Amer. Jour. Math.*, vol. 47, July, 1925.

Chapter XV.—EVANS, Economics and the Calculus of Variations, *Proc. Nat. Acad. Sci.*, vol, 11, Jan., 1925. EVANS, The Mathematical Theory of Economics, *Amer. Math. Monthly*, vol. 32, March, 1925. HOTELLING, A General Mathematical Theory of Depreciation, *Jour. Amer. Statistical Assoc.* Sept., 1925. ROOS, Dynamical Economics, *Proc. Nat. Acad. Sci.*, vol. 13, March, 1927. ROOS, A Dynamical Theory of Economics, *Jour. Polit. Econ.*, vol. 35, Oct., 1927. ROOS, A Mathematical Theory of Depreciation and Replacement, *Amer. Jour. Math.*, vol. 50, Jan., 1928. ROOS, The Problem of Depreciation in the Calculus of Variations, *Bull. Amer. Math. Soc.*, vol. 34, March-April, 1928. ROOS, Generalized Lagrange Problems in the Calculus of Variations, *Trans. Amer. Math. Soc.*, vol. 30, April, 1928. ROOS, Some Problems of Business Forecasting, *Proc. Nat. Acad. Sci.*, vol. 15, March, 1929. PIXLEY, A Theory of Discontinuous Price and Production Curves, *Amer. Jour. Math.* (in press). SCHULTZ, Mathematical Economics and the Quantitative Method, *Journ. Polit. Econ.*, vol. 35, *Oct.*, 1927.

APPENDIX II

ECONOMICS AND THE CALCULUS OF VARIATIONS

1. Some indication of a rather general theory may easily be given. It is natural to think of situations in economics as problems of maxima and minima. Presumably this is not the only way, since such problems lead themselves to determining equations of various types—algebraic, differential, integral, etc.—and it may therefore be possible to relate the statement of the theory to these equations more directly. Such formulations were instanced in Chapter IV where they were based on the notion of offer and demand and were perforce inadequate and of limited application. Even if these indications could be made more general and more unified, it would still be reasonable to construct a systematic analysis of the sort suggested by Chapters XIV and XV, since the idea of striving towards maxima is suggested by so many of the processes of business life. This analysis can be made to cover all the problems so far treated.

In order to make the ideas clear we must introduce the notion of functional, of which the quantity Π of the last two chapters is an instance. The Π, being an integral, depends on all the values of the integrand from $t = t_0$ to $t = t_1$; its value is determined by the function $p(t)$, say, as a whole, and not by the value of $p(t)$ at any specific time. Combinations of integrals yield again quantities of this same general type. We shall say that Π is a functional of $p(t)$ if, given the values of $p(t)$ for t in a certain interval (t_0, t_1), the value of Π is determined, and indicate this relation in the customary manner

$$\Pi = \Pi[p(t)_{t_0}^{t_1}]. \tag{1}$$

It may happen that the functional will not be defined for all functions $p(t)$, but merely for those which possess a derivative. A case in point would be the total variation function used in the theory of foreign exchange. Although of course the derivative of a function is determined when the function is given, and therefore mention of the derivative as well as the function may be

deemed unnecessary, it is convenient to emphasize the character of the functional by some explicit notation. Thus

$$\Pi[p(t),\ p'(t)_{t_0}^{t_1}]$$

will indicate a quantity which is defined when $p(t)$ has a derivative $p'(t)$ in the interval $(t_0,\ t_1)$; it is a specialization of the kind of functional indicated by (1). Similarly, the functional may involve parameters $x,\ y,\ \ldots$ as variables in the usual sense, and may have some reference to one or more particular values of t, as in the case of the quantities

$$\int_{t_0}^{t_1} K(t_2,\ t)p(t)dt$$
$$p(t_2) + \int_{t_0}^{t_1} R(x,\ t_2,\ t)p(t)dt$$

which involve in a special manner the value $t = t_2$. The meaning of such a symbol as

$$F[p(t),\ p'(t),\ p''(t)_{t_0}^{t_1};\ q(t)_{t_0'}^{t_1'};\ x,\ y,\ t_2,\ t_3]$$

is now clear.

If the functional Π or F is an integral, and we wish to make it a maximum, we are involved in a problem of the calculus of variations. In the case of more general functionals the problem is one in the calculus of functionals, but since again the same general methods are applicable it is desirable to retain the earlier and more suggestive name. Thus in Chapter XIV we discussed a routine problem of the calculus of variations, where the first derivative of the unknown function was involved (a problem of class 1, as it may be called), while in Chapter XV we discussed problems in which the functional was defined without reference to such derivatives (problems of class 0).

2. We may regard the economic system as divided into various compartments, according to convenience; a compartment may consist of the services of many individuals, *e.g.*, manual labor in a certain place, or the individual may at the same time figure in several different compartments, *e.g.*, with labor in one and with some form of capital in another. The problem is to discuss the flow of commodities or services through these various compartments. Let these compartments be numbered $1,\ 2,\ \ldots,\ n$, and let $\dot{x}_i$ be the rate at which the specific commodity or service i is produced in the compartment $[i]$. Let $\dot{X}_i$ be the rate at which this commodity issues from $[i]$, $r_i(t)$ the amount of

it which is present at time t in $[i]$, and $\dot{r}_i(t) = dr_i/dt$. We have

$$\dot{x}_i - \dot{X}_i = \dot{r}_i \tag{2}$$

With regard to exchanges among the compartments, let $\dot{x}_i^{(j)}$ be the rate at which the commodity or service i is transferred from $[i]$ to another compartment $[j]$, and $\dot{r}_i^{(j)}$ the portion of this which is not used on the production of the commodity or service j. If $\dot{X}_i^{(j)}$ denotes the portion of $\dot{x}_i^{(j)}$ which is used for the production of j, we have

$$\dot{X}_i = \sum_{j=1}^{n} \dot{x}_i^{(j)} = \sum_{j=1}^{n} (\dot{X}_i^{(j)} + \dot{r}_i^{(j)}). \tag{3}$$

We may let $r_i^{(j)}$ represent the accumulated amount of i in $[j]$, so that $\dot{r}_i^{(j)} = dr_i^{(j)}/dt$.

Further let

$$\varphi_k[\dot{x}, {}^{t_{11}}_{t_{01}}, \cdots], k = 1, 2, \cdots, m$$

and

$$\pi_\rho[\dot{x}, {}^{t'_{11}}_{t'_{01}}, \cdots], \rho = 1, 2, \cdots \sigma,$$

be functionals involving all or some of the variables just mentioned, and their derivatives. In these the rates $\dot{x}_i^{(j)}$, etc., may be regarded as entering as factors of production or rates of consumption, or as components of prices, etc., while the r_i, $r_i^{(j)}$, etc., enter as capitals.

A general system of economics may now be regarded as given by the following conditions:

$$\varphi_k = 0, k = 1, 2, \cdots, m, \tag{4}$$

$$\delta_{\dot{x}_{\rho_1}} \ldots \Pi_\rho = 0, \rho = 1, 2, \cdots, \sigma. \tag{5}$$

where the subscripts $\dot{x}_{\rho_1}, \ldots$ of the variation sign δ indicate the quantities which are allowed to vary, in making Π_ρ a maximum or minimum; that is to say, $\dot{x}_{\rho_1}, \ldots$ are replaced by $\dot{x}_{\rho_1} + \epsilon_{\rho_1}\psi_{\rho_1} \ldots$, and (5) has the significance

$$\sum_i \epsilon_{\rho i}\left(\frac{\partial \Pi_\rho}{\partial \epsilon_{\rho i}}\right)_{\epsilon_{\rho i} = 0} \cdot \tag{5.1}$$

The functions which vary must remain subject to the relation (4).

The functional relations (4) may be regarded as furnishing equations of demand or describing technical relations of manufacture, involving the factors of production (somewhat in the nature of cooking recipes); the φ_k may be called functionals of manufacture. The functionals Π_ρ may be called characteristic

functionals for the system whose extremal positions will determine the variable functions of the system (they may be, say, one or more integrals with respect to t of weighted sums of squares of rates of production, or various total profits over an interval of time, or—sinister suggestion—the total profit for some privileged set of compartments, or some convenient sociological character). The presence of more than one characteristic function which has a special relation to one commodity indicates a competitive character in the corresponding compartments. Equations of type (4), when they refer to equations of demand, may, in the progress of systematization, be replaced by equations of the type (5).

There are frequent specializations of these functions of a systematic character. From the nature of $[i]$ it may happen that $\dot{x}_i^{(i)} = 0$, so that (3) may be written in the form

$$\dot{X}_i = \sum_{j=1}^{n}{}' \dot{x}_i^{(j)}. \tag{3.1}$$

For many compartments the amount of i which is required will be determined, and in such cases the equations (4) may be regarded as solvable for the corresponding $\dot{x}_i^{(j)}$, say in the form

$$\dot{x}_i^{(j)} - \dot{r}_i^{(j)} = \int_t^\infty \dot{x}_j(\tau) d_\tau f_i^{(j)}[\dot{x}_1 \cdot \cdot \cdot , \tau]. \tag{4.1}$$

This gives the amount of i which is required for j in terms of the projected output from $[j]$. Usually also the Π_ρ will be additive with respect to intervals of time so that they may be regarded as integrals. The functionals may also of course degenerate into ordinary functions.

3. Fundamental for most economic systems is the analysis of the compartments relating to money. For simplicity, let us consider these as a single one, the first, comprising money proper and bank deposits subject to check, thus imagining the banks and mints to form the compartment [1]. The equations which involve 1 as suffix or index will serve incidentally to define certain ratios, $p_1, \ . \ . \ . \ p_n$, called prices. Thus we have

$$\dot{x}_1 - \dot{r}_1 = \dot{X}_1 = \sum_{k=1}^{n} \dot{x}_1^{(k)} \tag{6}$$

$$\dot{x}_1^{(k)} = \dot{r}_1^{(k)} + \sum_{i=1}^{n} p_i \dot{x}_i^{(k)} = p_k \dot{x}_k + \dot{E}_k' \tag{7}$$

where E_k' is the flow of credit into $[k]$ not due to sale of k. The last member of the equation (7) may be regarded as resulting from elimination of the equations of credit. The rate of repayment of loans appears as one of the items of the middle member.

In this connection, by introducing the terms $\dot{e}_i$, the rate of transfer of money proper from $[i]$, and $\dot{E}_i$, the rate of transfer from $[i]$ of other forms of media of exchange, m_i, the amount of money proper, and M_i, the amount of other media of exchange in $[i]$ at time t, we have

$$\frac{d}{dt}(m_i + M_i) = \dot{r}_1^{(i)}$$

The velocities of circulation v_i and V_i of m_i and M_i, respectively, will be defined by the equations

$$m_i v_i = \dot{e}_i, \; M_i V_i = \dot{E}_i,$$

But, remembering (7),

$$\dot{x}_1^{(i)} - \dot{r}_1^{(i)} = \dot{e}_i + \dot{E}_i = \sum_1^n p_j \dot{x}_j^{(i)}.$$

Hence

$$m_i v_i + M_i V_i = \sum_{j=1}^n p_j \dot{x}_j^{(i)}.$$

If we sum these equations for all i, and introduce the averages of Chapter IX, we obtain the equation of exchange in the form

$$mv + MV = \sum_{j=1}^n \sum_{i=1}^n p_j \dot{x}_j^{(i)} = \sum_{j=1}^n p_j(\dot{x}_j - \dot{r}_j) = T.$$

4. The situation of Chapter XIV may be regarded as a special case of this analysis in the following way. Suppose that the system is stabilized except for some particular commodity, related to a single compartment [0], so that fluxes and prices which refer to other commodities may be regarded as constants, the disturbances of this commodity causing negligible perturbations in those others. If for [0] we introduce the equations (4) or (4.1) in the form

$$\dot{x}_0^{(j)} = \left(a_j p_0 + b_j + h_j \frac{dp_0}{dt}\right) x_j,$$

the a_j, b_j, h_j, being constants, we have

$$\dot{x}_0 - \dot{r}_0 = \sum_{j=1}^n x_0^{(j)} = \sum_{j=1}^n \left(a_j p_0 + b_j + h_j \frac{dp_0}{dt}\right) \dot{x}_0;$$

that is,

$$\dot{x}_0 - \dot{r}_0 = ap_0 + b + h\frac{dp_0}{dt}, \tag{8}$$

with a, b, h constants, $a = \Sigma_j a_j \dot{x}_j$, etc. If the suffix 1 refers to money, as before, the last two members of (7) yield the equation

$$\dot{r}_1^{(0)} = p_0\dot{x}_0 + \dot{E}_0' - \sum_{i=1}^{n} p_i\dot{x}_i^{(0)};$$

and if here again we simplify the possible relation (4), writing

$$\dot{x}_i^{(0)} = \dot{r}_i^{(0)} + f_i^{(0)}\dot{x}_0,$$

taking the $f_i^{(0)}$as linear in $\dot{x}_0$, and if we assume that $\dot{E}_0' = 0$, substitution in the above equation yields us the following:

$$\dot{r}_1^{(0)} = p_0\dot{x}_0 - (A\dot{x}_0^2 + B\dot{x}_0 + C). \tag{9}$$

This is essentially the situation of Chapter IX if we take as the characteristic function the quantity

$$\Pi_0 = r_1^{(0)}(t_1) - r_1^{(0)}(t_0) = \int_{t_0}^{t_1} \dot{r}_1^{(0)}(t)dt.$$

If we suppose that $h = 0$, the situation reverts to that of Chapter I. On the other hand if we consider the commodity as occurring in two or more compartments and define the characteristic functions as the partial profits in these compartments we are able to investigate problems of competition.

5. An exceedingly simple system, approximate to a Utopian monotony, is obtained by assuming that in the equations (4.1) merely constant multiples of the $\dot{x}_j$ appear, so that

$$\dot{x}_i - \dot{r}_i = \sum_{j=1}^{n} f_i^{(j)}\dot{x}_j + \sum_{j=1}^{n} \dot{r}_i^{(j)}.$$

These equations may be solved for the $\dot{x}_1, \ldots \dot{x}_n$, if the determinant

$$\begin{vmatrix} 1 - f_1^{(1)} & -f_1^{(2)} & \cdots & -f_1^{(n)} \\ \cdot & \cdot & \cdot & \cdot \\ -f_n^{(1)} & -f_n^{(2)} & \cdots & 1 - f_n^{(n)} \end{vmatrix}$$

does not vanish. Hence the Π_ρ may be expressed in terms of the various $r_i^{(j)}$ and their rates of change—unused capitals and services. However, in a system where waste products are utilized when capital is fully employed, this false simplicity disappears, because it is precisely then that the determinant vanishes—for we must

have $\dot{x}_j$ not all zero even when all the $\dot{r}_i$, $\dot{r}_i^{(j)}$ vanish. The extremal character of a Π_ρ which was given in terms of these hoardings would not then determine the variable functions. On the other hand, it is precisely in such a system that we can most easily condense the compartments into larger "compound" industries, and study first the large groups as compartments, then the parts of these, and so on, until the investigation becomes sufficiently minute.

The theory outlined here has been made somewhat more detailed by Prof. C. F. Roos in the memoirs cited in Appendix I. In particular, Professor Roos gives detailed consideration to problems of competition, and to more general problems (problems of depreciation and replacement) under various forms of organization, in which are involved the prices of present in terms of future income, that is, phenomena connected with the rate of interest. The initial paper in this latter direction is that of Professor Hotelling. Professor Roos considers also in some detail the choice of initial or limiting conditions—initial price, production, etc.,—with respect to the particular problems. It may be remarked here that these conditions should enter into the general theory by means of additional equations of the general type (4), with the special characteristic that they involve quantities which are arbitrary for the problem in question. These quantities may be arbitrary constants, such as initial prices or times, or they may be arbitrary functions, as in the problem of Sec. **83**, where y and p are assumed to be known before the time $t = t_0$.

In connection with the general theory, finally, it will be noticed that from another point of view considerations of integrability for variational equations in functionals may be introduced, in a manner which is analogous to the methods of Chapters XI to XIII.

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